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A Combinatorial Approach to Wealth Distribution in Multi–Aggregate Systems

ANDREA MONACO AND ADAMARIA PERROTTA

Abstract. We study a closed monetary system in which agents are organised into interacting aggregates. Starting from a combinatorial description of the accessible configurations of the system, we derive analytically the joint equilibrium distribution of aggregate wealth and aggregate size. The derivation follows from a counting argument combined with an entropy maximisation principle. In a continuous formulation the allocation of wealth among agents naturally leads to Dirichlet-type structures, which provide a convenient framework for describing aggregation effects. Numerical simulations of the stochastic dynamics confirm the analytical predictions and reproduce the theoretical distributions.

1. Introduction

The statistical characterization of wealth distribution has long attracted the attention of economists, mathematicians, and statisticians. Beyond its relevance for economic policy and social welfare, it raises a more fundamental scientific question: are there universal statistical laws governing the allocation of economic resources in large populations?

The first systematic empirical investigation is dated 1897 and it is commonly attributed to Vilfredo Pareto, who observed that the upper tail of income distributions follows a power law [14]. This empirical regularity, now known as Pareto's law [4, 5], suggests that the probability density of income x behaves asymptotically as

$$p(x) \propto x^{-(1+\alpha)},$$

for sufficiently large values of x.

More generally, the attempt to understand social and economic phenomena through statistical regularities dates back to the nineteenth century. Adolphe Quetelet, for instance, proposed the idea of a "social physics", arguing that collective properties of human societies may exhibit stable statistical patterns [17].

During the twentieth century several researchers attempted to explain wealth distributions using stochastic models. Robert Gibrat proposed that proportional random growth of incomes leads naturally to lognormal distributions, an idea known as Gibrat's Law [9]. Later contributions by Kalecki and Champernowne showed that stochastic processes can also generate Pareto-type tails, providing theoretical mechanisms capable of reproducing the empirical regularities observed by Pareto [11, 3].

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Most of these classical approaches adopt a "one-body" perspective in which the wealth of each individual evolves independently according to a stochastic growth process. While such models capture some empirical features, they do not explicitly account for the interactions among economic agents.

A different viewpoint emerged with the development of *econophysics*, which applies methods from statistical physics to economic systems [13, 21, 19]. In this framework the economy is viewed as a many-body system composed of interacting agents. A particularly influential example is the model proposed by Dragulescu and Yakovenko, in which agents randomly exchange money while the total amount of money in the system is conserved [7].

Several extensions of this kinetic exchange framework have been proposed, including models with saving propensity and heterogeneous interaction rules [2, 4, 1].

By analogy with the Boltzmann–Gibbs distribution of energy, this model predicts an exponential stationary distribution of money, a result that has received considerable empirical support for the bulk of income distributions in many countries [21].

Although kinetic exchange models [5, 16, 18] successfully reproduce several stylized facts of wealth distributions, they typically treat agents as identical units interacting only at the individual level. Real economic systems, however, display a clear hierarchical structure, where individuals belong to firms, firms to sectors, and sectors to larger economic systems.

In this work we extend the statistical-mechanics approach to wealth dynamics by introducing an additional level of organization. Instead of considering only individual agents, we study a closed monetary system composed of interacting *aggregates*, each representing a group of agents sharing common characteristics. This intermediate level allows us to capture both the microscopic dynamics of monetary exchanges and the mesoscopic evolution of groups within the system.

Our presentation complements our earlier study [12]. There, the emphasis was primarily on the statistical–mechanics viewpoint, the resulting equilibrium laws, and their validation against numerical simulations. In the present article we focus instead on the mathematical structure of the model and provide a more detailed derivation of the main results, with the aim of making the underlying arguments transparent to a mathematical audience.

Within this framework we investigate the joint statistical distribution of wealth and aggregate size and study how interactions across different levels of organization shape the macroscopic patterns of wealth allocation.

The novelty of the present work lies in the introduction of an intermediate aggregate level together with an explicit combinatorial derivation of the joint equilibrium distribution of wealth and aggregate size.

Our goal is not to reproduce the full complexity of real economies, but rather to identify the minimal statistical structure capable of generating the main empirical regularities of wealth distributions. In this sense the model can be viewed as an idealized laboratory system, in the spirit of statistical mechanics, for studying how patterns of economic inequality emerge from simple stochastic interactions.

2. The multi–aggregate monetary model

We now introduce the mathematical structure of the monetary system that will be studied in the remainder of the paper. The model can be viewed as an extension of the kinetic exchange framework introduced by Yakovenko [7], in which money is randomly exchanged between agents while the total amount of money in the system remains conserved.

We consider a closed monetary economy composed of D agents. Each agent i possesses a non–negative quantity of money m_i. The total amount of money in the system is fixed and equal to M, so that the conservation law

$$\sum_{i=1}^{D} m_i = M$$

holds at all times.

Agents are organised into N_a aggregates, denoted by A_k, for $k = 1, \ldots, N_a$. An aggregate represents a group of agents sharing common characteristics, such as belonging to the same firm, economic sector, or institutional community.

For each aggregate A_k, we denote by d_k the number of agents belonging to that aggregate. The quantity d_k therefore represents the *size* of the aggregate A_k. The conservation of the total number of agents implies that

$$\sum_{k=1}^{N_a} d_k = D.$$

The total monetary wealth of the k-th aggregate is defined as

$$M_k = \sum_{i \in A_k} m_i,$$

so that summing over all aggregates we recover the total monetary mass

$$\sum_{k=1}^{N_a} M_k = M.$$

The state of the system at a given time can therefore be described by the set of pairs

$$(M_k, d_k), \qquad k = 1, \ldots, N_a,$$

which represent respectively the wealth and the size of each aggregate A_k.

The dynamics of the system is driven by two classes of stochastic interactions.

Monetary exchange between agents.
Two agents i and j are randomly selected and exchange a quantity Δm of money according to the rule

$$(m_i, m_j) \to (m_i - \Delta m,\, m_j + \Delta m), \qquad m_i + m_j = (m_i - \Delta m) + (m_j + \Delta m).$$

This mechanism conserves the total wealth of the system.

Aggregate interaction.
Aggregates themselves may interact through the exchange of agents. If Δn agents move from aggregate A_k to aggregate A_ℓ, the sizes of the two aggregates change according to

$$(d_k, d_\ell) \to (d_k - \Delta n,\, d_\ell + \Delta n), \qquad d_k + d_\ell = (d_k - \Delta n) + (d_\ell + \Delta n).$$

This mechanism conserves the total number of agents.

Both interaction mechanisms preserve the global quantities M and D. As a consequence, the system evolves within the set of configurations satisfying the constraints

$$\sum_{k=1}^{N_a} M_k = M, \qquad \sum_{k=1}^{N_a} d_k = D.$$

In analogy with statistical mechanics we assume that, in equilibrium, all microscopic configurations compatible with these conservation laws are equiprobable. Under this hypothesis the equilibrium properties of the system can be derived through a combinatorial counting argument.

In particular, we will determine the joint probability distribution of aggregate wealth and aggregate size. This distribution characterizes the statistical structure of the system at equilibrium and will be derived explicitly in the next section.

3. Derivation of the equilibrium distribution

In this section we derive the joint equilibrium distribution of aggregate wealth and aggregate size within the framework introduced in the previous section. We prove the following:

Theorem 3.1. *In the context of the framework set out in Section 2 and in terms of the number of aggregates N_a, the total wealth M, and the total number of agents D, the joint equilibrium distribution $p(m,d)$ of aggregates wealth m and aggregate size d is given by*

$$p(m,d) = \binom{m+d-1}{d-1} \frac{N_a M^m (D-N_a)^{d-1}}{(D+M)^{m+d}}. \tag{1}$$

The corresponding marginal distributions of aggregate size $p(d)$ and of aggregate wealth $p(m)$ are given by

$$p(d) = \frac{N_a}{D^d}(D-N_a)^{d-1} \tag{2}$$

and

$$p(m) = \frac{N_a}{M}\left(\frac{M}{M+N_a}\right)^{m+1} \tag{3}$$

Proof. Let $n_{m,d}$ denote the number of aggregates having monetary wealth m and size d. The quantities $n_{m,d}$ therefore describe the macroscopic configuration of the system.

Because the total number of aggregates, agents and money are fixed, the following conservation relations must hold:

$$\sum_{m,d} n_{m,d} = N_a, \qquad \sum_{m,d} d\, n_{m,d} = D, \qquad \sum_{m,d} m\, n_{m,d} = M.$$

These relations represent the macroscopic constraints of the system.

We now compute the number of microscopic configurations compatible with a given macrostate $\{n_{m,d}\}$. First, the N_a aggregates must be distributed among the possible states (m,d), which can be done in

$$\frac{N_a!}{\prod_{m,d} n_{m,d}!}$$

different ways. In addition, for each aggregate characterised by size d and wealth m, the quantity m must be distributed among d agents. The number of such configurations is

$$\binom{m+d-1}{d-1}.$$

Taking these contributions together, the total number of microscopic configurations associated with the macrostate $\{n_{m,d}\}$ is

$$W(n) = \frac{N_a!}{\prod_{m,d} n_{m,d}!} \prod_{m,d} \binom{m+d-1}{d-1}^{n_{m,d}}.$$

The quantity $W(n)$ counts the number of microscopic configurations of the system corresponding to the macroscopic state $\{n_{m,d}\}$. Under the assumption that all microscopic configurations compatible with conservation constraints are equally likely, the probability of observing a given macrostate is proportional to $W(n)$. Consequently, the equilibrium configuration corresponds to the maximum of $W(n)$, or equivalently to the entropy

$$S = \ln W(n).$$

Using Stirling's approximation

$$\ln(n!) \approx n \ln n - n,$$

which is valid for large n, we obtain

$$S = N_a \ln N_a - N_a - \sum_{m,d} n_{m,d} \ln n_{m,d} + \sum_{m,d} n_{m,d} + \sum_{m,d} n_{m,d} \ln \binom{m+d-1}{d-1}.$$

The equilibrium configuration corresponds to the maximum of the entropy subject to the conservation constraints. Introducing Lagrange multipliers α, β and γ we impose

$$\delta \left(S - \alpha \sum d n_{m,d} - \beta \sum m n_{m,d} - \gamma \sum n_{m,d} \right) = 0.$$

Taking the derivative with respect to $n_{m,d}$ and setting it equal to zero gives the equilibrium condition and leads to

$$n_{m,d} = \binom{m+d-1}{d-1} e^{-\alpha d} e^{-\beta m} e^{-\gamma}.$$

Dividing by the total number of aggregates we obtain the joint equilibrium distribution

$$p(m,d) = C \binom{m+d-1}{d-1} e^{-\beta m} e^{-\alpha(d-1)} \tag{4}$$

where C is the normalization constant into which the term involving γ has been absorbed.

We now determine the parameters C, α and β by imposing the normalization of the probability distribution together with the macroscopic constraints. The normalization condition reads

$$\sum_{m=0}^{\infty} \sum_{d=1}^{\infty} C \binom{m+d-1}{d-1} e^{-\beta m} e^{-\alpha(d-1)} = 1.$$

To simplify this expression we compute the marginal distribution of aggregate size,

$$p(d) = \sum_{m=0}^{\infty} C \binom{m+d-1}{d-1} e^{-\beta m} e^{-\alpha(d-1)}.$$

Using the generating function identity

$$\sum_{m=0}^{\infty} \binom{m+d-1}{d-1} x^m = \frac{1}{(1-x)^d},$$

valid for $|x| < 1$, and setting $x = e^{-\beta}$ we obtain

$$p(d) = C e^{\alpha} \frac{e^{-\alpha d}}{(1-e^{-\beta})^d}.$$

Similarly, summing over d yields the marginal distribution of aggregate wealth

$$p(m) = C e^{\beta} \frac{e^{-\beta(m+1)}}{(1-e^{-\alpha})^{m+1}}.$$

Returning to the normalization condition we obtain

$$C \sum_{d=1}^{\infty} e^{\alpha} \frac{e^{-\alpha d}}{(1-e^{-\beta})^d} = 1.$$

This geometric series converges provided that

$$0 < \frac{e^{-\alpha}}{1 - e^{-\beta}} < 1.$$

Evaluating the series gives

$$C \frac{1}{1 - e^{-\alpha} - e^{-\beta}} = 1. \tag{5}$$

We next impose the constraint on the average aggregate size

$$\sum_{d=1}^{\infty} d\, p(d) = \frac{D}{N_a}.$$

Using the identity

$$\sum_{k=1}^{\infty} k x^k = \frac{x}{(1-x)^2}$$

we obtain

$$C \frac{1 - e^{-\beta}}{(1 - e^{-\alpha} - e^{-\beta})^2} = \frac{D}{N_a}. \tag{6}$$

Finally we impose the conservation of the average wealth

$$\sum_{m=0}^{\infty} m\, p(m) = \frac{M}{N_a}.$$

Repeating the same steps we obtain

$$C \frac{e^{-\beta}}{(1 - e^{-\alpha} - e^{-\beta})^2} = \frac{M}{N_a}.$$

Introducing the variables $x = e^{-\alpha}$, $y = e^{-\beta}$ and $z = C$ the problem reduces, by Eqs 4, 5, 6, to the system

$$\begin{cases} \dfrac{z}{1 - x - y} = 1 \\ \dfrac{(1-y)z}{(1-x-y)^2} = \dfrac{D}{N_a} \\ \dfrac{yz}{(1-x-y)^2} = \dfrac{M}{N_a} \end{cases}$$

whose solution is

$$x = \frac{D - N_a}{D + M}, \qquad y = \frac{M}{D + M}, \qquad z = \frac{N_a}{D + M}.$$

Substituting these expressions we finally obtain

$$C = \frac{N_a}{D + M}, \qquad \alpha = -\ln\left(\frac{D - N_a}{D + M}\right), \qquad \beta = -\ln\left(\frac{M}{D + M}\right).$$

If we substitute these values in $p(m, d)$, $p(d)$ and $p(m)$, we get 1, 2 and 3.
Note that

$$e^{-\alpha} + e^{-\beta} = \frac{D - N_a}{D + M} + \frac{M}{D + M} = 1 - \frac{N_a}{D + M} < 1,$$

thus the convergence condition is always satisfied for systems satisfying $N_a < D$. $\square$

4. Numerical validation

In this section we provide a numerical validation of the analytical results derived in the previous sections. The purpose of the simulations is not to study the full dynamical behaviour of the system, but rather to verify that the stochastic evolution of the model leads to the equilibrium distributions predicted by the theoretical analysis.

4.1. **Simulation setup.**

We simulate a closed monetary system composed of D agents organised into N_a aggregates. Each agent initially possesses the same amount of money, while aggregates are assigned an initial size distribution consistent with the constraint

$$\sum_{k=1}^{N_a} d_k = D.$$

The total monetary mass of the system is fixed and equal to M,

$$\sum_{i=1}^{D} M_i = M,$$

so that the system evolves under the same conservation laws assumed in the theoretical derivation.

The dynamics of the simulation follows the stochastic rules introduced in Section 2. At each step, we randomly select: two agents from the whole population of aggregates, and an amount of money Δm to be exchanged. If the amount of money Δm exceeds the wealth of the randomly selected agents, a new value for Δm is sampled. In addition, aggregates may interact through the exchange of a random number of agents Δn. Both mechanisms preserve the total number of agents and the total amount of money.

The simulation is iterated for a sufficiently large number of steps so that the system reaches a stationary statistical regime. Once this regime is reached we sample the quantities (m_k, d_k) associated with each aggregate in order to reconstruct the empirical distributions of aggregate wealth and aggregate size.

In the numerical experiments we consider systems with $D = 10^4$ agents and $N_a = 500$ aggregates and evolve the system for approximately 10^7 interaction steps before sampling the stationary distributions.

4.2. **Comparison with theoretical predictions.**

The theoretical analysis developed in Section 3 predicts that the joint equilibrium distribution of aggregate wealth and aggregate size is given by Eq. (4), where the parameters C, α and β depend only on the macroscopic quantities M, D and N_a.

From this joint distribution, we obtain the marginal distributions of aggregate wealth and aggregate size. These theoretical predictions can then be compared with the distributions obtained from the simulated data.

Figure 1 shows the empirical distributions obtained from the numerical simulation together with the corresponding theoretical curves. The agreement between the two is very strong for both quantities. In particular, the distribution of aggregate wealth exhibits the exponential decay predicted by the analytical model, while the aggregate size distribution follows the theoretical law derived in Section 3.

This comparison confirms that the stochastic dynamics of the system, despite being defined at the level of individual interactions, converges towards the equilibrium distribution predicted by the statistical–mechanical analysis.

As an additional consistency check we consider the limiting case in which the system contains a single aggregate. In this situation the model reduces to the original kinetic exchange framework studied by Yakovenko [7]. The simulations reproduce the exponential wealth distribution predicted in that setting, providing a further validation of the numerical implementation.

Overall, the numerical experiments support the analytical results derived in the previous sections and confirm that the proposed multi–aggregate framework preserves the main statistical properties of the original kinetic exchange model while introducing an additional level of organisation in the system.

5. Conclusions

In this work we have studied the statistical properties of a closed monetary system in which agents are organised into interacting aggregates. Starting from simple conservation principles for the total number of agents and the total monetary mass, we derived the joint equilibrium distribution of aggregate wealth and aggregate size.

The derivation is based on a combinatorial description of the accessible configurations of the system. Assuming that all microscopic configurations compatible with the macroscopic constraints are equiprobable, the equilibrium distribution follows from a standard entropy maximisation argument [10, 15]. The resulting probability law, derived in Section 3 (see Eq. (4)), depends only on the macroscopic quantities that characterise the system, namely the total wealth, the number of agents, and the number of aggregates.

A central aspect of the analysis is the combinatorial factor appearing in the equilibrium distribution, which counts the number of possible ways in which wealth can be distributed among the agents belonging to a given aggregate. This term provides the mathematical bridge between the statistical mechanics formulation of the model and classical results from combinatorics and probability theory, making explicit how macroscopic statistical laws emerge from the enumeration of admissible microscopic configurations.

When the allocation of wealth is considered in a continuous framework, the same combinatorial structure naturally leads to Dirichlet-type distributions [6, 8]. The aggregation properties of the Dirichlet family offer a convenient mathematical framework for describing how individual wealth variables combine when agents are grouped into larger units, providing a direct connection between microscopic allocation mechanisms and the statistical behaviour of aggregates.

The analytical results derived in the paper have been validated through numerical simulations of the stochastic dynamics. The simulated distributions of aggregate wealth and aggregate size are in strong agreement with the theoretical predictions obtained from the combinatorial analysis, confirming that the stochastic evolution of the system reproduces the equilibrium laws derived analytically. This comparison is illustrated in Figure 1.

From a mathematical perspective, the model illustrates how relatively simple conservation constraints, when combined with elementary counting arguments, can generate non–trivial statistical laws in systems composed of many interacting agents.

Several mathematical extensions of the present framework appear natural. One may consider interaction mechanisms in which exchange rates depend explicitly on aggregate characteristics, or investigate alternative conservation constraints. Another promising direction concerns continuous formulations of the model, where Dirichlet structures and related families of probability distributions may play an even more central role in describing aggregation phenomena.

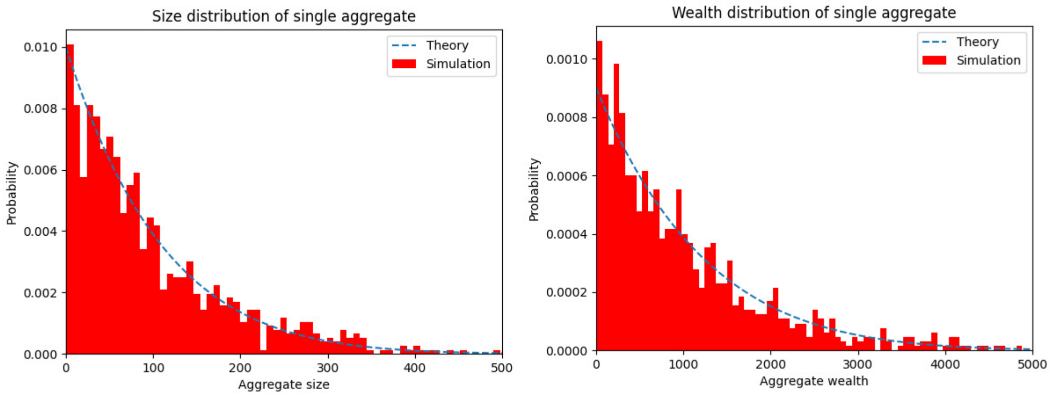

FIGURE 1. Numerical validation of the equilibrium distributions derived in Section 3. The figure compares the empirical distributions obtained from the stochastic simulations with the theoretical predictions of the combinatorial model (see Eq. (4)). The left panel shows the distribution of aggregate size, while the right panel shows the distribution of aggregate wealth. In both cases the simulated data are in strong agreement with the analytical distributions, confirming that the stochastic dynamics of the system converges to the equilibrium law derived from the combinatorial analysis.

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Andrea Monaco is Assistant Professor in Financial Mathematics at University College Dublin, where he serves as Program Director for the MSc in Financial Mathematics within the School of Mathematics and Statistics. He holds a Ph.D. in Physics from Université Grenoble Alpes. He spent over 14 years in the financial sector, specializing in pricing and risk assessment for banks and insurance companies, before returning to academia in 2021. Research interests: Risk Management, Wealth Dynamics and Machine Learning for Finance.

Adamaria Perrotta is Assistant Professor in Financial Mathematics at University College Dublin, where she serves as Program Director for the BSc in Financial Mathematics within the School of Mathematics and Statistics. She holds a Ph.D. in Mathematics from University of Naples Federico II. She spent over 5 years in the financial sector, specializing in pricing and risk assessment for banks and insurance companies, before returning to academia in 2016. Research interests: Risk Management, Wealth Dynamics and Machine Learning for Finance.

(Both authors) School of Mathematics and Statistics, University College Dublin
E-mail address: andrea.monaco@ucd.ie, adamaria.perrotta@ucd.ie