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IRISH MATHEMATICAL SOCIETY NEWSLETTER NUMBER 10

# IRISH MATHEMATICAL SOCIETY

# NEWSLETTER

NUMBER 10 MARCH 1984

## THE IRISH MATHEMATICAL SOCIETY



## NEWSLETTER

EDITOR

*Donal Hurley*

ASSOCIATE EDITOR

*Patrick Fitzpatrick*

The aim of the *Newsletter* is to inform Society members about the activities of the Society and also about items of general mathematical interest. It appears three times each year: March, September and December. Deadline for copy is six weeks prior to publication date.

The *Newsletter* also seeks articles of mathematical interest written in an expository manner. All parts of mathematics are welcome, pure and applied, old and new.

Manuscripts should be typewritten and double-spaced on A4 paper. Authors should send two copies and keep one copy as protection against possible loss. Prepare illustrations carefully on separate sheets of paper in black ink, the original without lettering and a copy with lettering added.

Correspondence relating to the *Newsletter* should be sent to:


Irish Mathematical Society Newsletter,
Department of Mathematics,
University College,
Cork.

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## IRISH MATHEMATICAL SOCIETY

### Ordinary Meeting

Wednesday, April 18th, 1984, 12.15 p.m., DIAS

Agenda

1. Minutes of Ordinary Meeting of 21st. December 1983.
2. Matters Arising.
3. Proposal to change Constitutional procedures for elections to membership.
4. Proposal to change rule on lapsing of membership.
5. Reciprocity with I.M.T.A.
6. Elections to ordinary membership.
7. Any other business.

### Ordinary Meeting

Wednesday 22nd December, 1983, 12.15 p.m., DIAS

1. There were 16 members present. The President, A.G. O'Farrell, took the Chair. The minutes of the Ordinary meeting of 31/3/83 were read and signed.

2. The President reported that, having written to the Minister for Education concerning post-graduate awards for those with a B.A. in Mathematical Sciences from N.U.I., he had received a negative reply. However, as a result of a private conversation he had with the Minister, the President proposed to write again reiterating the points of the first letter and requesting that a mathematician be appointed to the Committee which oversees these awards.

3. The Treasurer, G. Enright, presented his report which had been audited by Professors J.J.H. Miller and D.J. Simms. The meeting approved the report and expressed its appreciation to the Treasurer for his careful handling of the financial affairs of the Society.

The President mentioned that the Committee of the I.M.S. had agreed to:

(i) the use of a form by those members seeking support for Conferences (available from the Treasurer);

(ii) a library subscription rate of £15 for the Newsletter.

4. At the request of the President, the meeting ratified the membership of all those listed in the membership list who have not been formally elected to membership.

The meeting expressed its thanks to the Treasurer for preparing the list.

5. The President reported on the progress of continuing negotiations by M. Clancy (on behalf of I.M.S.) with the I.M.T.A. concerning reciprocal membership.

6. The Secretary enquired from Aer Lingus about the possibility of the I.M.S. sponsoring a prize for mathematical projects at the Young Scientists' Exhibition. The reply, eventually, was that the organisers felt that a prize would not be sufficient to stimulate interest in Mathematical projects, an interest which is at a low ebb. Having discussed various ideas, including the possibility of circulating a list of possible topics for projects, the Committee felt that the most effective way the I.M.S. could use its limited resources would be to concentrate on a small number of schools. D. Hurley agreed to work with M. Brennan (Bower School, Athlone) and R. Timoney with H. Macklin (Blackrock College). All members were asked to try and stimulate interest in the Young Scientists' Exhbition.



During discussion A. Wood mentioned that there would be a course on Mathematical Modelling for teachers on 3rd March 1984 at N.I.H.E. (Dublin) run in conjunction with the I.M.T.A. He felt that this kind of mathematics would lend itself to project work.

7. The following were nominated, seconded and elected:

| | |
|---|---|
| Secretary: | R. Timoney (T.C.D.) |
| Treasurer: | G. Enright (M.I.C.E.) |
| Committee: | R. Bates (Met. Service) |
| | P. Fitzpatrick (U.C.C.) |
| | M. Stynes (Waterford R.T.C.) |
| | S. Tobin (U.C.G.) |
| | N. Buttimore (T.C.D.) |

all for two years, except N. Buttimore, who is elected for one year. The President, A.G. O'Farrell, the Vice-President, F. Holland, P. Boland, M. Clancy and J. Hannah continue on the Committee until December 1984.

8. A. Solomon raised the question of human rights for Jewish Mathematicians in the Soviet Union. As a result of intense pressure from academics and societies in the West, the situation has improved somewhat in that the practice of removing doctorates from persons applying to emigrate seems to have been discontinued. However, discrimination continues and he mentioned the examples of Scharansky, Josef Begun (both in prison), Brailovsky (in exile) and Jewish entrants to Moscow University.

9. N.N. Yanenko was nominated for membership of the Society by J. Miller, seconded by D. Simms (in writing). After some discussion it was agreed to follow the constitutional procedure and vote on his candidature at the next Ordinary Meeting.

10. The President reported that the Committee had nominated J. Hannah as the I.M.S. representative to the National Committee for Mathematics of the R.I.A.

Also the Secretary is to write to the N.B.S.T. expressing the gratitude of the I.M.S. for their printing of the Newsletter.

*R. Timoney (Secretary)*

## MASSERA CAMPAIGN

A motion of support for the Soviet mathematician/statistician V. Kipnis and for the Uruguayan mathematician J.L. Massera was passed at the December 1982 ordinary meeting of the Irish Mathematical Society. At the same meeting, a voluntary collection for the Support Massera Campaign was collected and resulted in £15 being forwarded to the headquarters of the campaign in Toronto, Canada.

During the year, the secretary of the Society wrote to the Minister for Foreign Affairs Mr P. Barry, the Uruguayan Ambassador in Berne Senor Grambruno, and Academician A.P. Aleksandrov (Moscow) concerning the situation of these mathematicians. No replies were received from Berne or Moscow, but the Minister for Foreign Affairs was quite sympathetic. Mr Barry was already aware and concerned about Dr Kipnis' situation and promised to inform the Society of any information regarding this case which might become available through the Irish Embassy in Moscow. As Ireland has no diplomatic relations with Uruguay, he was unable to obtain information regarding Professor Massera's case, but promised to take such action as is possible, through, for instance, the United Nations Commission on Human Rights.

In the last year, we have had no new information regarding V. Kipnis and so must assume that his situation has not impr-



oved in any way. The latest bulletin regarding Professor Massera (June 1983) tells us that he is now (in his 78th year) a symbol of resistance to his fellow prisoners and a symbol of guilt to his tormentors who have said "Massera will stay in prison until he dies." His torture continues.

Our small efforts on their behalf may never yield the results desired - freedom to emigrate for both Kipnis and Massera (Massera has been offered university positions in both France and Italy), but our involvement can only help and encourage those in the forefront of the campaign to serve them and help prevent such situations from arising in the future.

*S. Dineen*

## MEMBERSHIP LIST SUPPLEMENT 84-1, 12th January 1984

Ms N. Sheehan, 10 Oak Park Road, Carlow

Mr L. Leyden, Regional Technical College, Waterford

Dr B. Goldsmith, Dublin Institute of Technology, Kevin St.

Mr V. Ryan, 6 Kerley Road, Model Farm Road, Cork.

Prof. F. Harary, University of Michigan, U.S.A.

Prof. R. Geoghegan, State University of New York, U.S.A.

Dr R. Critchley, N.I.H.E., Limerick.

Mr M. Ryan, N.I.H.E., Dublin.

Mr M. O'hEigeartaigh, N.I.H.E., Dublin.

Ms A. Brady, Student, U.C.D.

Mr T. McGrane, Student, U.C.D.

## CONFERENCE SPONSORSHIP

The Irish Mathematical Society has a small fund out of which it can give limited assistance to the organisers of mathematical conferences.

Application forms are available from the Treasurer.

*G. Enright (Treasurer)*

## INVITATION TO NOMINATE SPEAKERS

## AT

## CONGRESS 1986

Acting on an invitation from the International Mathematical Union to prepare a panel of mathematicians who will be invited to address the 1986 International Congress of Mathematicians at Berkeley, California, the National Committee for Mathematics hereby solicits names of suitable speakers. Each nomination should be properly motivated and should include a short list of publications. All submissions should be in the hands of the undermentioned by April 30, 1984.

*Secretary,*
*National Committee for Mathematics,*
*Royal Irish Academy,*
*19 Dawson Street,*
*Dublin 2.*



# NEWS AND ANNOUNCEMENTS

## THE BOOLE PAPERS

Late in November, 1983, University College Cork became aware of the following item to be auctioned in Sotheby's (London) on 8 December:

> *Boole (George, 1815-1864), mathematician and logician, Fellow of the Royal Society). Large collection of papers by and relating to Boole assembled by his sister Mary Ann Boole, including the manuscripts and typescripts of her (unpublished) biography of him, her copies of letters by him and of his (unpublished) poems and lectures on mythology, education, astronomy and Ireland, together with some of his autograph poetical drafts, autograph mathematical notes and a notebook, some two hundred or more autograph letters by him to his sister and other letters sent to him, thousands of pages, in a tin box, sold as a collection not subject to return £500-600.
>
> *Boole, who was born and educated at Lincoln and pursued a distinguished academic career at Queen's College Cork, wrote some fifty books and papers on logic and mathematics of which the most important and "durable" is his *Laws of Thought* (1854) - "a work of astonishing originality and power" (*Dictionary of National Biography*).

Naturally, it was felt that the College's Boole Library would be an eminently suitable home for this unique collection of material and an immediate effort was made to raise funds - £600 being regarded as a rather crude under-estimate, considering Boole's international reputation. Through the generosity of the College, the Library, the Royal Irish Academy, Cork Chamber of Commerce and a number of interested individuals, a large amount of money was pledged.

The bidding at Sotheby's was carried out on behalf of the College by the firm of Bernard Quaritch who, on the previous day, had paid the world record price of £8 million for a 12th century illuminated German manuscript on behalf of the German Government. In the event, the College acquired the Boole papers with the more modest bid of £2,400 (sterling) despite some transatlantic opposition. Outside the library of the Royal Society in London, Cork now has the largest collection of Booleana in the world and the Boole Library should certainly become a centre for Boolean studies.

The collection is a very extensive and important one which throws great light on the personal aspects of Boole's life, particularly his life in Cork. It consists mostly of personal letters to and from his family covering the period 1845-1855, but there are a number of items of mathematical interest. For example, there is a copybook written when he was about 16, containing a large number of worked examples from Gregory's *Examples on the Differential Calculus*. Each exercise has been carefully worked out by Boole and neatly written into the copybook with loving care. When doing questions on differentiation, Boole always uses the notation if $y = x^3$ then $dy = 3x^2dx$. There is also a notebook with jottings on geometry, elementary number theory and word games such as "change BLACK to WHITE in the minimum number of moves".

The collection also contains a number of original offprints of Boole's early papers, but the main item of mathematical interest is an unpublished manuscript on astronomy in which Boole uses probability to make various predictions such as the occurrence of binary stars. It is hoped that the various items in the collection, mathematical and non-mathematical will be collected together in book form to give a personal and down-to-earth view of the life and times of George Boole.

Boole's sister Mary Ann, who assembled the collection, also had personal connections with Cork. She was governess to the children of William Fitzgerald, Church of Ireland Bishop of Cork,



who later became Bishop of Killaloe. One of those children, whom Mary Ann taught, afterwards became a famous scientist. He was George Francis Fitzgerald (1851-1901) who was well known for his work in electro-magnetic theory and one of the forerunners of Albert Einstein. Fitzgerald's name is commemorated in "Lorentz-Fitzgerald contractions" in Physics.

Where the Boole papers have lain hidden since Mary Ann's death in 1882 is a mystery and the sellers remain frustratingly anonymous. However, perhaps there were some omens - Sotheby's codename for the auction was "SEAMUS" and the sale took place on the 119th anniversary of Boole's death.

*D. MacHale*

Editorial Note: D. MacHale's biography of G. Boole is due to appear shortly.

## PERSONAL ITEMS

*Dr Berthold Franzen* has been awarded a Department of Education Post-Doctoral Fellowship at the Dublin Institute of Technology, Kevin Street. His research interests are in the area of Abelian groups and module theory.

*Dr Brendan Goldsmith* of the Mathematics Department, Dublin Institute of Technology, Kevin Street, has been appointed Head of Department.

*Dr Andrew Pressley* of the Mathematics Department, Trinity College, Dublin, will be on leave at the Mathematics Institute in Berkeley, California, from March to September 1984.

*Dr Brian Smyth* of the Mathematics Department, University College, Dublin, has been appointed to a professorship at Notre Dame University, Indiana, U.S.A.

*Dr James Ward* who had a temporary position at University College, Cork, has taken up a Junior Lectureship in Mathematics at University College, Galway.

## CRACKING A RECORD NUMBER

*Mathematicians Solve a Three-Century-Old Puzzle in 32 Hours*

Sandia National Laboratories in Albuquerque is a sprawling research establishment best known for its work on highly secret defense projects, including nuclear weaponry. Last week Sandia exploded a different sort of bombshell. Its mathematicians announced that they had factored a 69-digit number, the largest ever to be subjected to such numerical dissection. Their triumph is more than an intellectual exercise. It could have far-flung repercussions for national security.

As anyone who has ever passed through intermediate algebra knows (or once knew), factoring means breaking a number into its smallest whole-number multiplicands greater than 1. For example, 3 and 5 are the only such factors of 15. But as numbers get larger, factoring them becomes increasingly difficult. Until recently, mathematicians despaired of factoring any number above 50 digits. They calculated that it would take the fastest computer, performing as many as a billion divisions a second, more than 100 million years to finish the task.

Then, in the fall of 1982, a chance encounter closed the gap. During a scientific conference in Winnipeg, Canada, Gustavus Simmons, head of Sandia's applied-math department, was mulling the factoring problem over a few beers with another mathematician and an engineer from Cray Research, makers of the world's fastest computer. The engineer, Tony Warnock, pointed out that the internal workings of the Cray were especially suited to factoring, which is essentially done by a process of trial and error. Unlike ordinary computers, the Cray



can sample whole clusters of numbers simultaneously, like a sieve sifting through sand for coins.

At Sandia, Simmons joined with his colleagues Mathematicians James Davis and Diane Holdridge to teach their own Cray how to factor. That involved developing an algorithm, or set of algebraic instructions, that would break the problem down into small steps. They succeeded admirably. In rapid succession they factored numbers of 58, 60, 63 and 67 digits.

At this point, however, even the power of their Cray seemed to have reached its limit. But the Sandia team made one more try. This time their target was the last unfactored number in a famous list compiled by the 17th century French mathematician Marin Mersenne. The number:132686104398972053177608575-506090561429353935989033525802891469459697, which mercifully can be expressed as $2^{251} - 1$. After a total of 32 hr. and 12 min. of computer time, snatched at odd hours over a period of a month, they came up with their answer. Mersenne's number had three basic factors: 178230287214063289511 and 616768821-98695257501367 and 12070396178249893039969681. Says Simmons: "You can't help feeling triumphant after solving a problem that has been around more than three centuries."

Some may not share in the jubilation, especially if they are dependent on a widely used cryptographic system thought to be uncrackable. Known as RSA (the initials of its three inventors), it employs difficult-to-factor multidigit numbers to encode secrets and keep then secure. These include electronic funds transfers and military messages. By factoring the numbers, the codes can be broken. When RSA was first proposed, its inventors suggested using 80-digit numbers on the assumption that they were too big to be factored. Obviously, with researchers at Sandia closing in on ever larger numbers, even RSA could eventually fall to the code breakers.

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From *TIME* Magazine (February 13th, 1984)

## THE SUPERBRAIN COMPETITION AT U.C.C.

Over coffee after a student mathematics society meeting at University College, Cork, a discussion arose as to who were the best mathematical students in College. Students of Electrical Engineering claimed that because of high points requirements, they were obviously the best. However, Science students hotly disputed this. A challenge went out which led to the organizing of a competitive examination open to all full-time registered students of College, regardless of subjects or faculty. It was dubbed the Superbrain Competition and the questions were set and corrected by Dr D. MacHale. So as not to give an advantage to students who had taken advanced courses, the topics were those of the Honours Leaving Certificate course though, of course, the standard was a good deal more difficult. Prizes were kindly donated by Arthur Guinness and Company.

Out of 44 entrants, top place was filled by a fourth year Science student, Stephen Buckley, with a score of 70%. However, the next eight places were filled by students of Electrical Engineering, led by James Cunnane and Barry Ambrose, with scores of 66, 58, 57, 42, 41, 38, 36 and 34. A Science student filled tenth place with 33. Further down the scale, marks were 32, 31, 30, 29 (3), 28, 27, 26, 25 (2), 24 (3), 23, 22, 21 (3), 19, 17, 16 (2), 15, 14, 13, 8 (2), 7, 4, 3 (2). Interestingly the bottom three places were also filled by Engineering students! The average mark was thus about 25 and the pass mark was declared to be 0 - taking part being equivalent to passing!

There were many ingenious solutions and suggestions but weaknesses appeared on topics such as Geometry, Diophantine Equations, Induction, and the Associative Law. Many of the students found themselves labouring at a mathematics examination for the first time. It was comforting to note that a girl came in sixth position with an excellent score of 41. Put that in



your pipe and smoke it!

## U.C.C. SUPERBRAIN 1984

Answer any *ten* questions

---

1. Using each number once and once only, place the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9 in a 3x3 square so that all rows, columns and diagonals sum to eight *different* totals, i.e. no two sums are equal.

2. If x and y are positive integers, find all solutions of the equation

$$2xy - 4x^2 + 12x - 5y = 11.$$

3. If ABC is a triangle show, with proof, how to find points S and R on the line BC, P on the line AB, and Q on the line AC, such that PQRS is a square.

4. Evaluate the indefinite integral $\int (\sec^9\theta + \sec^8\theta)d\theta$.

5. Ten books are arranged in a row on a shelf. In how many different ways can this be done, if one particular book A must always be to the left of another book B?

6. Assuming that $\lim_{x\to 0} \sqrt[x]{\frac{1+x}{1-x}}$ exists, find its value.

7. If $n = 2^k$ for $k \geq 1$, show that ${}^nC_r = \binom{n}{r}$, the number of combinations of n things r at a time, is an even number, for $1 < r < n$.

8. If $A = \{a,b,c,d\}$ is a set of four distinct elements, is it possible to define a closed binary operation $*$ on A such that the associative law $x*(y*z) = (x*y)*z$ *never* holds for any triple $x,y,z \in A$, equal or distinct?

9. If $\alpha = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \ldots$, $\beta = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \ldots$,

$\gamma = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \ldots$, assuming that all three series converge for $x \in R$, prove that $\alpha^3 + \beta^3 + \gamma^3 = 1 + 3\alpha\beta\gamma$.

10. Prove that $\cos 29^\circ$ is not a rational number.

11. Form a nine digit number using each of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, once and only once so that the number formed by the nine digits is a multiple of nine, the number formed by the first eight digits is a multiple of eight, the number formed by the first seven digits is a multiple of seven, and so on, i.e. the number formed by the first $n$ digits is a multiple of $n$, for $1 \leq n \leq 9$.



## LETTER TO THE EDITOR

December 1983

Dear Editor,

The National Committee for Mathematics received the following letter from the International Mathematical Union appealing for financial support for the Special Development Fund, and would appreciate your bringing it to the notice of your readers in the belief that many of them would wish to contribute to the fund. Donations can be sent anytime either directly to the banks mentioned in Professor Lehto's letter or to the undersigned, marked "I.M.U. Special Development Fund".

Yours sincerely,

*Secretary,*
*National Committee for Mathematics,*
*Royal Irish Academy,*
*19 Dawson Street,*
*Dublin 2.*

May 4, 1983

TO ALL NATIONAL COMMITTEES FOR MATHEMATICS

The Special Development Fund aids IMU to fulfill the important obligation of helping developing countries within the framework of mathematical research. The means of the Fund, which go unreduced to mathematicians from developing countries, are used primarily for travel grants to young mathematicians, to make them possible to participate in International Congresses of Mathematicians. The Executive Committee of IMU elects an international committee to distribute the grants.

Means to the Special Development Fund come from private donations. This letter is addressed to you in the hope that you could make a contribution to the Fund, either directly or by making an appeal among the mathematical community of your country. Donations can be sent at any time and in any convertible currency, to the following accounts:

Schweizerische Kreditanstalt
Stadtfiliale Zurich-Rigiplatz
Universitätstrasse 105
CH-8033 Zürich, Switzerland
Account Number 0862-656208-21

Kansallis-Osake-Pankki
Aleksanterkatu 42
SF-00100 Helsinki 10, Finland
Account Number 100020-411-USD-5705 FR.

The next goal is to collect money for travel grants for the 1986 International Congress of Mathematicians in Berkeley.

With best thanks for your cooperation,

Yours sincerely,

*Ollie Lehto*



# THE EVOLUTION OF RESONANT OSCILLATIONS IN CLOSED TUBES

*E.A. Cox[1] and M.P. Mortell[2]*

## 1. INTRODUCTION

This paper discusses the formulation and solution of a nonlinear initial value, boundary value problem that arises from a simple experiment in gas dynamics. A tube which is closed at one end, contains a gas. The gas in the tube is driven by an oscillating piston. It is observed that when the frequency of the piston is near to a natural frequency of the tube the resulting gas motion is periodic and characterised by a shock wave travelling over and back along the tube. The theoretical work to explain the final periodic motion goes back to Betchov [2] and Chester [3]. The reader should consult Seymour and Mortell [9] for more recent work on the problem. However, the problem of the evolution of the periodic motion of the gas from an initial state has not until now [4] been solved.

It is worthwhile noting, at this juncture, that nonlinear effects, such as shocks, can occur without any dramatically large input into the system. For example, in the present case when the piston is operating at the fundamental frequency of the tube, a shock has been observed even though the ratio of piston displacement to tube length is of the order $10^{-2}$ [8].

Before giving the details of the particular problem, the broader background in which it is set will be sketched. The study of nonlinear waves began with the pioneering work of Stokes [10] and Riemann [7]. Whitham [11] distinguishes two main classes of waves, hyperbolic and dispersive waves. Hyperbolic waves are solutions of a set of hyperbolic partial differential equations and our problem fits into this class. The intersection of characteristics for a nonlinear hyperbolic equation gives rise to the physical phenomenon of a shock.

If the nonlinear wave is travelling in one direction only and into a constant state, the exact solution is called a 'simple wave' and was known to Riemann. Riemann also exposed the fundamental difficulty when nonlinear waves are travelling in opposite directions. It is not, in general, possible to integrate the equations for the characteristics. This corresponds to the fact that nonlinear waves interact and one must know the details of the right-going waves to calculate how the left-going wave will propagate, and *vice-versa*. The fundamental difficulty still remains, and even in such authoritative works as [11] and [6] problems of nonlinear waves travelling through each other in opposite directions receive scant attention.

The problem discussed in this paper involves waves in a tube of finite length and, since shocks appear, automatically involves the propagation of nonlinear waves through each other in a finite space domain. The problem will be approached through a novel use of perturbation methods.

Finally it should be noted that at resonance, aside from the appearance of a shock which is a nonlinear phenomenon, linear theory predicts the evolution to an unbounded motion.

## 2. FORMULATION

In terms of nondimensional Lagrangian variables the equations expressing conservation of mass and linear momentum for the gas are

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial u}{\partial x} = 0 \qquad (2.1)$$

$$\frac{\partial u}{\partial t} + a^2 \frac{\partial \rho}{\partial x} = 0 \qquad (2.2)$$

where $u$, $\rho$, $a$, denote gas velocity, density, and sound speed at the gas particle $x$, for time $t$. The equation of state for the isentropic flow of an ideal gas may be written as:



$$a^2 = \rho^{\gamma-1} \qquad (2.3)$$

where $\gamma$ is the gas constant.

Equations (2.1), (2.2) and (2.3) are supplemented by the boundary conditions

$$u(0,t) = 0$$

$$u(1,t) = -2\pi\epsilon\omega \sin 2\pi\omega t, \quad \epsilon << 1 \qquad (2.4)$$

and by the initial conditions

$$\begin{aligned} u(x,t) &= 0 \qquad 0 \leq x \leq 1, \qquad t \leq 0, \\ a(x,t) &= 1 \qquad 0 \leq x \leq 1, \qquad t \leq 0. \end{aligned} \qquad (2.5)$$

The problem is to follow the evolution of the gas motion under the prescribed boundary conditions (2.4) from the initial undisturbed state (2.5) to the final periodic state.

Replacing $\rho$ in (2.1) and (2.2) by using (2.3) the resulting equations can be combined to form the coupled system

$$\begin{aligned} \left[\frac{\partial}{\partial t} + a^{\frac{\gamma+1}{\gamma-1}} \frac{\partial}{\partial x}\right]\left[u + \frac{2}{\gamma-1}a\right] &= 0 \\ \left[\frac{\partial}{\partial t} - a^{\frac{\gamma+1}{\gamma-1}} \frac{\partial}{\partial x}\right]\left[u - \frac{2}{\gamma-1}a\right] &= 0 \end{aligned} \qquad (2.6)$$

The Riemann Invariant $u + \frac{2}{\gamma-1}a$ is constant on the characteristic curves $\alpha(x,t)$ = constant given by

$$\left.\frac{\partial x}{\partial t}\right|_{\alpha} = -a^{\frac{\gamma+1}{\gamma-1}} \qquad (2.7)$$

with a similar statement for the other Riemann Invariant. Equation (2.7) cannot be integrated since $a(x,t)$ is unknown until the solution is found.

The approach adopted to solve the system (2.6) is to assume a regular perturbation expansion for u and a in (2.6) of the form

$$u(x,t) = \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \dots$$
$$a(x,t) = 1 + \varepsilon a_1(x,t) + \varepsilon^2 a_2(x,t) + \dots \qquad (2.8)$$

Linear theory results from terms at $O(\varepsilon)$ with a nonlinear correction at $O(\varepsilon^2)$. The linear terms $u_1, a_1$ satisfy

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right]\left[u_1 + \frac{2}{\gamma-1} a_1\right] = 0$$
$$\left[\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right]\left[u_1 - \frac{2}{\gamma-1} a_1\right] = 0 \qquad (2.9)$$

with general solution

$$u_1 = f(t+x-1) - g(t-x)$$
$$a_1 = -\frac{(\gamma-1)}{2}[f(t+x-1) + g(t-x)] \qquad (2.10)$$

where f,g are arbitrary functions - the linear Riemann Invariants. It should be noted that the two sets of linear characteristic curves t+x-1 = constant and t-x = constant are parallel straight lines and are independent of the solution $u_1, a_1$. In other words the linear waves neither distort, nor interact with each other.

The boundary conditions on x=0, x=1 given by equations (2.4) imply that on x=1

$$f(t) - f(t-2) = -2\pi\varepsilon\omega\sin 2\pi\omega t, \quad t > 0 \qquad (2.11)$$

an equation which is augmented by the initial condition

$$f(t) = 0, \quad t \leq 0 \qquad (2.12)$$



When $\omega = \frac{n}{2}$, n = 1,2,3 ... equation (2.11) predicts that

$$f(t) \text{ is asymptotic to } -nt\sin(n\pi t), \text{ as } t \to \infty, \qquad (2.13)$$

in other words predicting unbounded growth frequencies equal to the natural frequencies of the gas tube.

On substituting for $u_1$ and $a_1$ into (2.6), (2.8) we obtain the particular integral

$$u_2(x,t) = \frac{\gamma+1}{2} x[g(t-x)g'(t-x) + f(t+x-1)f'(t+x-1)]$$
$$+ \frac{\gamma+1}{4} x[f'(t+x-1)G(t-x) - g'(t-x)F(t+x-1)]$$

where

$$G(t) = \int^t g(y)dy \quad \text{and} \quad F(t) = \int^t f(y)dy.$$

We note that the complementary functions associated with $u_2$ may be absorbed into the representation for $u_1$.

The novel feature of the approach now outlined is that the boundary conditions are applied not to $u_1$ and $u_2$ separately but to the combined approximation $\varepsilon u_1 + \varepsilon^2 u_2$, i.e.

$$\varepsilon u_1(0,t) + \varepsilon^2 u_2(0,t) = 0 \qquad (2.15)$$

and

$$\varepsilon u_1(1,t) + \varepsilon^2 u_2(1,t) = -2\pi\varepsilon\omega\sin 2\pi\omega t.$$

The aim is to formulate in one relationship a mechanism of controlling the linear growth by the nonlinear terms.

The boundary condition on x = 0 implies that

$$f(t-1) = g(t), \quad t \geq 0 \qquad (2.16)$$

After some manipulation the boundary condition on x = 1 implies that f(t), the linear Riemann Invariant, satisfies the nonlinear equation

$$-2\pi\omega \sin 2\pi\omega t = f(t)-f(t-2) + \varepsilon\frac{(\gamma+1)}{2}[f(t)f'(t)+f(t-2)f'(t-2)]$$

$$+ \varepsilon\frac{(\gamma+1)}{4}[-f'(t-2)\int^{t} f(y)dy + f'(t)\int^{t-2} f(y)dy \qquad (2.17)$$

with initial condition (2.12).

Equation (2.17) is a nonlinear functional differential equation of neutral type, see [5]. The equation of linear theory is included in (2.17): the nonlinear terms in (2.17) which are in the brackets associated with $\frac{(\gamma+1)}{2}$ represent the effect of amplitude dispersion by which shocks form, while the remaining term represents the nonlinear interaction of opposite travelling waves.

The solution of the nonlinear initial value, boundary value problem on the semi-infinite strip $0 \leq x \leq 1$, $t \geq 0$ and defined by equations (2.1) - (2.5) has now been reduced to a solution of the nonlinear equation (2.17) with the initial conditions (2.12). When the linear Riemann Invariant, f, is known, the particle velocity and sound speed in the tube can be found from the representations (2.10), (2.16).

## 3. GOVERNING PARTIAL DIFFERENTIAL EQUATION

The functional differential equation (2.17) was derived using only a regular perturbation expansion and retaining the sum of the first two terms as the basic approximation. We now show how equation (2.17) can be simplified to a hyperbolic partial differential equation by the use of a two variable expansion technique. There are two natural time scales in the physical problem under consideration: the time for a signal to travel the length of the tube and the time for a shock to form. The basic assumption underlying the simplification of (2.17) is that the latter time scale is much larger than the former. The fast time scale is $t^+ = t$ and the slow time scale is $\tilde{t} = \varepsilon t$. The function f(t) is then expanded in the form



$$f(t;\varepsilon) = f_1(t^+,\tilde{t}) + \varepsilon f_2(t^+,\tilde{t}) + .. \qquad (3.1)$$

Since the primary motivation is to find solutions near the resonant frequency $\omega = \frac{1}{2}$, we introduce the small detuning parameter

$$\Delta = 2\omega - 1 << 1. \qquad (3.2)$$

We now seek solutions which are periodic in the fast time variable $t^+$ with the same period as the piston, viz, $1/\omega$ and are slowly modulated on the long time scale.

Therefore we assume that

$$f_i(t^+ - \frac{1}{\omega},\tilde{t}) = f_i(t^+,\tilde{t}) \qquad (3.3)$$

On using (3.1) - (3.3) in the functional differential equation (2.17) we obtain the partial differential equation

$$2\varepsilon\frac{\partial f_1}{\partial \tilde{t}} + \frac{\Delta}{\omega}\frac{\partial f_1}{\partial t^+} + \varepsilon(\gamma+1)f_1\frac{\partial f_1}{\partial t^+} = -2\pi\varepsilon\omega \sin 2\pi\omega t^+ \qquad (3.4)$$

where terms involving $O(\Delta^2)$, $O(\varepsilon\Delta)$, $O(\varepsilon^2)$ have been neglected. We note that the integral terms in (2.17) which represent the interaction of oppositely travelling waves are $O(\varepsilon\Delta)$ and thus negligible.

The initial condition corresponding to (2.12) and the state of rest is

$$f_1(t^+,0) = 0 \qquad (3.5)$$

Since the solution of (3.4) is periodic in $t^+$ with periods $\frac{1}{\omega}$, integration of (3.4) over a time interval of length $\frac{1}{\omega}$, with an appeal to weak shock conditions when necessary (see [11]), yields the mean condition

$$\int_0^{\frac{1}{\omega}} f_1(s,\tilde{t})\,ds = 0 \qquad (3.6)$$

Thus the mean value of f remains constant on lines of constant $\tilde{t}$ as the signal evolves. In order to put (3.4) in a form more amenable for analysis we define

$$F(\eta,\tau) = (\gamma+1)\varepsilon\omega f_1(t^+,\tilde{t}) + \Delta \qquad (3.7)$$

where

$$\eta = \omega t^+, \quad \tau = \frac{\tilde{t}}{2\varepsilon} \qquad (3.8)$$

Then (3.4) becomes

$$\frac{\partial F}{\partial \tau} + F\frac{\partial F}{\partial \eta} = -A\sin(2\pi\eta) \qquad (3.9)$$

$$A = 2\pi\varepsilon\omega^2(\gamma+1) << 1. \qquad (3.10)$$

The initial condition becomes

$$F(\eta,0) = \Delta \qquad (3.11)$$

The remainder of this paper is concerned with the analysis of (3.9) subject to (3.11). It should be noted that the physical properties of the system are all contained in the similarity parameter A, given by (3.10). Variations in the piston amplitude, $\varepsilon$, and frequency $\omega$, or the gas properties $\gamma$, corresponding to different experiments are immaterial to the solution of (3.12) as long as the parameter A remains constant.

## 4. EXACT SOLUTION

The nonlinear partial differential equation (3.9), which describes the evolution on the boundary x = 1 of the linear Riemann Invariant, is hyperbolic and can thus be studied by the method of characteristics. The transport equation



$$\frac{dF}{d\tau}(\eta,\tau) = -A\sin 2\pi\eta \qquad (4.1)$$

describes the variation of the signal F on the characteristic curves $\alpha(\eta,\tau)$ = constant given by

$$\frac{d\eta}{d\tau} = F(\eta,\tau) \qquad (4.2)$$

The characteristic curves are parameterised by $\alpha(\eta,0) = \eta$. The coupled system (4.1), (4.2) and the initial condition (3.11) are equivalent to the second order equation

$$\frac{d^2\eta}{d\tau^2} = -A\sin 2\pi\eta \qquad (4.3)$$

with initial conditions

$$\eta(0) = \alpha, \quad -\tfrac{1}{2} \le \alpha \le \tfrac{1}{2} \qquad (4.4)$$

and

$$\frac{d\eta}{d\tau}(0) = \Delta.$$

Thus the characteristic paths are given by the nonlinear pendulum equation (4.3) with the signal profile given by (4.1).

On using (4.2), equation (4.3) is written as

$$F\frac{dF}{d\eta} = -A\sin 2\pi\eta \qquad (4.5)$$

Integration of (4.5) yields

$$\left(\frac{d\eta}{d\tau}\right)^2 = F^2 = \frac{\beta^2}{4\pi^2}\{1 - m^2\sin^2(\pi\eta)\} \qquad (4.6)$$

where

$$m^2(\alpha,0) = \frac{1}{\sin^2\pi\alpha + \dfrac{\pi\Delta^2}{2A}}, \quad \beta^2 = \frac{8\pi A}{m^2} \qquad (4.7)$$

In a standard manner, integration of (4.6) then yields exact solutions for $\eta(\alpha,\tau)$ expressed in terms of elliptic functions (see [1]). With F given by (4.6) the solution of (3.9) can then be tabulated.

## 5. SOLUTION CURVES AND DISCUSSION

Fig. 1 corresponds to A = 0.01, Δ = 0.02 and shows the growth of the amplitude of the signal over the initial periods (as predicted by linear theory), but with a simultaneous cumulative distortion of the signal shape until a shock forms in the seventh cycle of the piston. One can see in this output what was anticipated in applying the boundary conditions as in (2.15).

Fig. 2 shows how the signal settles down to the periodic state, containing a shock, after about 30 cycles. Figs 3 and 4 show the evolution of the signal for the case A = 0.01, Δ = 0.06. This is a particularly interesting case since experiments show that the periodic state is continuous and is, in fact, essentially determined by linear theory. The figures show how a shock forms and then decays out of the system so that the eventual steady state is shockless. The solution thus goes through a nonlinear regime but eventually reaches a periodic state which is closely approximated by linear theory.

A shock is a dissipative mechanism so that when the piston is operating at or near resonance the shock dissipation balances the energy buildup due to the phase matching of the input and response to allow a final periodic state containing a shock. Away from resonance, e.g. when Δ = 0.06, there is a sufficient mismatch of phase to obviate an energy buildup and a shock cannot be sustained. Thus a continuous periodic steady state results.

The analysis given here is based on numerical calculations of the exact solution. It is also instructive to consider the phase plane associated with (4.3), (4,4) and the reader will find this in [4].



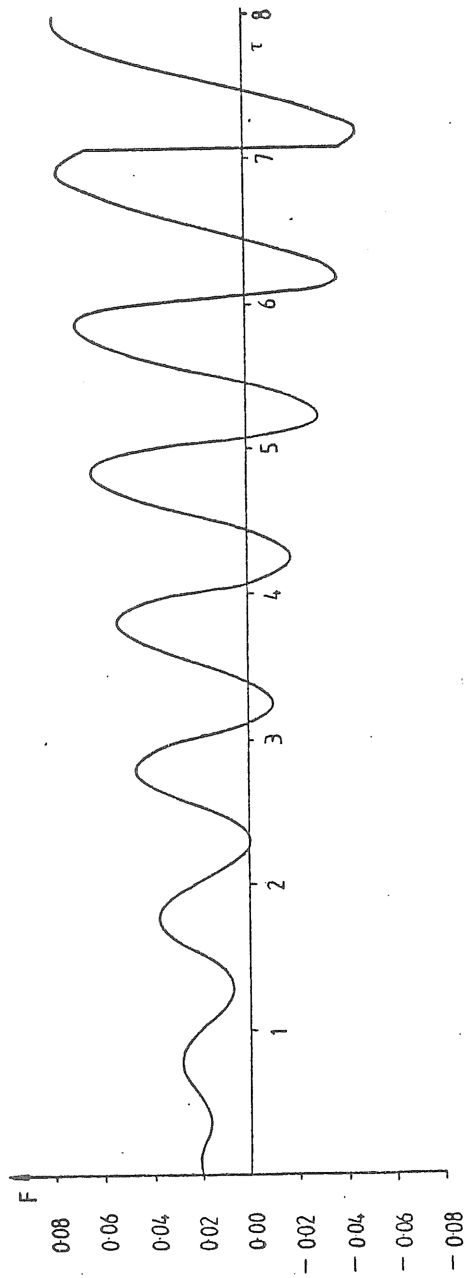


FIGURE 1

10.

11.

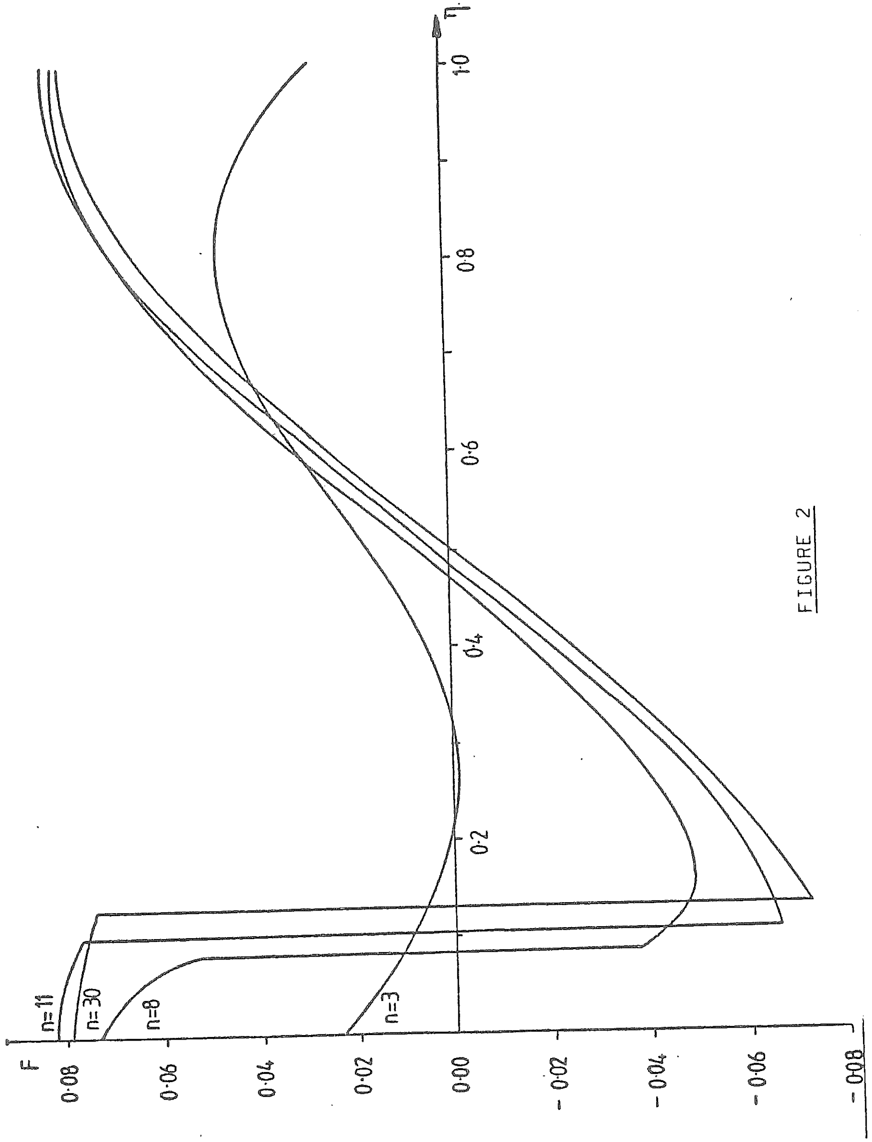


FIGURE 2



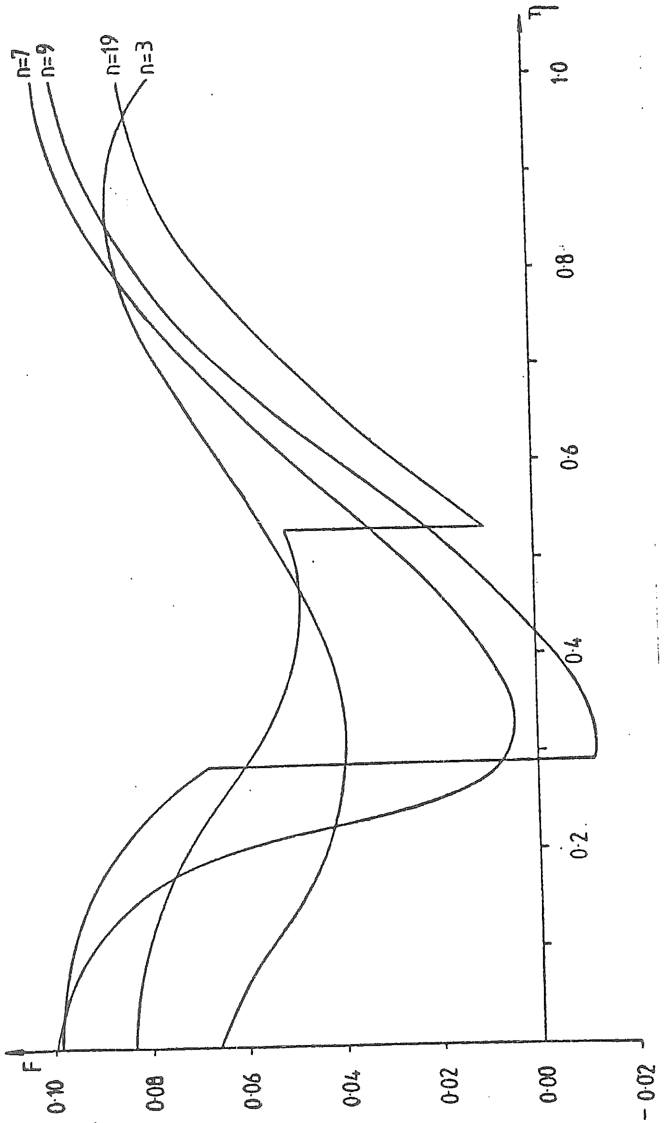


FIGURE 3

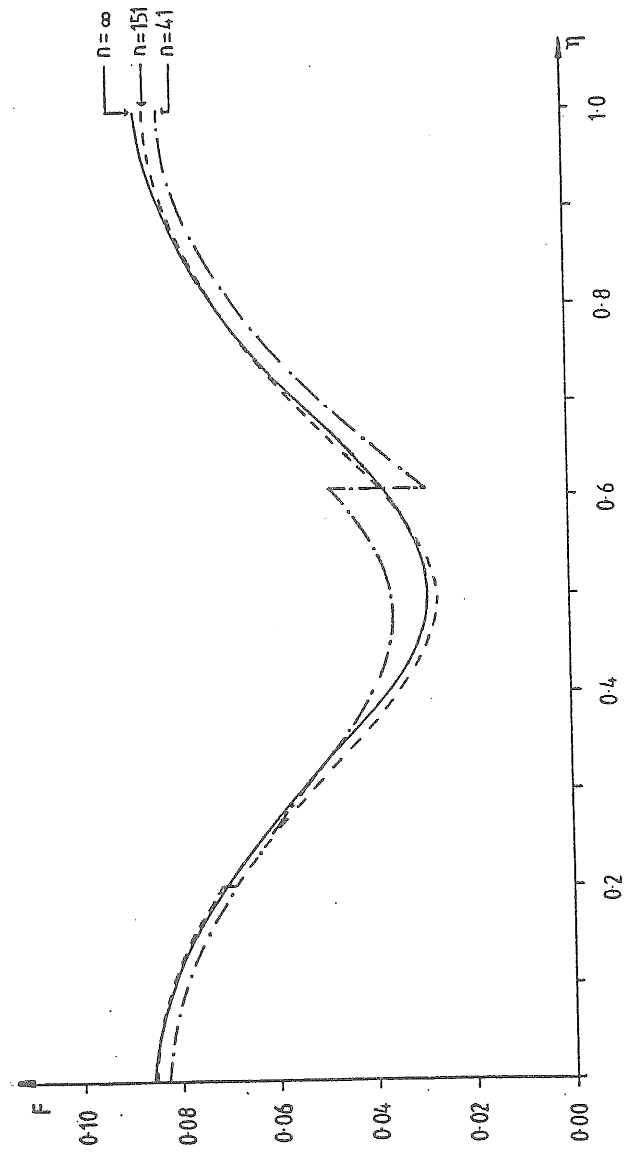


FIGURE 4



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# ISOTROPIC TENSORS AND SYMMETRIC GROUPS

*Rex Dark and Martin Newell*

## Introduction

Students usually meet tensors in courses on Mathematical Physics [1, Chapter 3], and are sometimes told that all isotropic tensors can be expressed as sums of products of Kronecker's $\delta$ and the alternating tensor $\varepsilon$ (see below). Several years ago, the first author was asked by an inquisitive student how to prove this fact. The author in question did not, at that time, know of a proof in the literature, and (having more energy then than now) he tried to find his own proof. He was led to a question about the group rings of symmetric groups, which he found interesting in its own right, and which he answered to the satisfaction of both himself and the student who had prompted the question. Soon afterwards, he learned that the statement about isotropic tensors is equivalent to a result proved by Weyl [4, page 53, Theorem (2.9.A)] using a different method. More recently, an error was noticed in the calculations in the group ring of the symmetric group, and we give here an exposition of a corrected version of this proof.

## Notation

For each positive integer m, put $\underline{m} = \{1,2, \ldots ,m\}$, and let $\underline{n}^{\underline{m}}$ be the set of maps from $\underline{m}$ to $\underline{n}$. An element of $\underline{n}^{\underline{m}}$ can be identified with a *polyindex* $(i) = (i_1,i_2, \ldots ,i_m)$ where $1 \le i_r \le n$ $(1 \le r \le m)$. With respect to given axes in n-dimensional Euclidean space $E^n$, a *tensor* u can be defined as a map from $\underline{n}^{\underline{m}}$ to the field of real numbers: for each polyindex $(i) \in \underline{n}^{\underline{m}}$, we have a real coordinate $u(i)$. We shall say that u has *dimension* n and *order* m.

Examples A tensor v of order 1 is the same as a *vector* in $E^n$ with coordinates $v(1), v(2), \ldots ,v(n)$. Similarly, a tensor T or order 2 can be regarded as an nxn *matrix* with entries



$T(i_1,i_2)$. In particular, we put

$$\delta(i_1,i_2) = \begin{cases} 1 & \text{if } i_1 = i , \\ 0 & \text{if } i_2 \neq i , \end{cases}$$

which corresponds to the identity matrix. Moreover, for each polyindex $(i) = (i_1,i_2, \ldots ,i_n) \in \underline{n}^{\underline{n}}$, we can define

$$\varepsilon(i) = \begin{cases} 1 & \text{if (i) is an even permutation of } \underline{n}, \\ -1 & \text{if (i) is an odd permutation of } \underline{n}, \\ 0 & \text{if (i) is not a permutation;} \end{cases}$$

then $\varepsilon$ is a tensor of order n, equal to its dimension. If T is a tensor of order 2, then we can use $\varepsilon$ to give a formula for its *determinant*:

$$\det T = \sum_{(i)\in \underline{n}^{\underline{n}}} \varepsilon(i_1,i_2, \ldots ,i_n)T(1,i_1)T(2,i_2)\ldots T(n,i_n).$$

Remark The indices $i_1$, $i_2$, ... ,$i_m$ are usually written as subscripts or superscripts, but we prefer to avoid double suffices; we consider only perpendicular axes in $E^n$, so we do not need to distinguish between cogredient and contragredient indices. Also, we shall not use the Summation Convention [3, page 59]. Examples of physical quantities represented by the concept are the *strain tensor*

$$e_{ik} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\right)$$

in the Theory of Elasticity [1, page 99, Section 3.101], and the *metric tensor* $g_{ik}$ of Differential Geometry (as used in the Theory of Relativity) which determines the incremental distance

$$ds = \sqrt{\sum_{i,k} g_{ik}dx^i dx^k} ;$$

in our case $g_{ik} = \delta(i,k)$.

*Definition:* When we change the axes in $E^n$, there is an nxn matrix $T(h,k)$ such that the coordinates of a vector v are altered to

$$v'(h) = \sum_{k=1}^{n} T(h,k)v(k)$$

For u to be a tensor of order m, it is required that its coordinates become

$$u'(i_1, \ldots, i_m) = \sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} T(i_1,j_1)\ldots T(i_m,j_m)u(j_1,\ldots,j_m)$$

A tensor is said to be *isotropic* if its coordinates remain unaltered when we change from one set of perpendicular axes to any other such set with the same orientation; this means that $u'(i) = u(i)$ whenever the matrix $T(h,k)$ is orthogonal with determinant 1.

Examples  It can be verified that the tensors $\delta$ and $\varepsilon$ defined above are both isotropic [1, page 87, Section 3.03], and it is known that, in the 3-dimensional case, any isotropic tensor of order 2 or 3 is a multiple of $\delta$ or $\varepsilon$ respectively. Moreover, it can be shown [1, page 88, Section 3.031] that the isotropic tensors of dimension 3 and order 4 are the linear combinations of the tensors

$$\delta(i_1,i_2)\delta(i_3,i_4), \quad \delta(i_1,i_3)\delta(i_2,i_4), \quad \delta(i_1,i_4)\delta(i_2,i_3)$$

which are called *outer products* of $\delta$ [1, page 115]. These facts can be used to motivate the next definition.

Notation

Taking outer products of $\delta$ and $\varepsilon$, we define the following isotropic tensors of dimension n and order m:

$$d(i) = \begin{cases} \delta(i_1,i_2)\delta(i_3,i_4) \ldots \delta(i_{m-1},i_m) & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd,} \end{cases}$$

$$e(i) = \begin{cases} \varepsilon(i_1,i_2, \ldots, i_n)d(i_{n+1}, \ldots, i_m) & \text{if } m \geq n, \\ 0 & \text{if } m < n. \end{cases}$$

If $\alpha$ is an element of the symmetric group $S_m$ of all per-



mutations of $\underline{m}$, and if $h\alpha$ denotes the image under $\alpha$ of an element $h \in \underline{m}$, then $\alpha$ acts on the polyindices (i) as a place permutation: $(\alpha i) = (i_{1\alpha}, i_{2\alpha}, \ldots, i_{m\alpha})$. This leads to an action on the tensors of order m given by $(u\alpha)(i) = u(\alpha i)$; we shall call the tensor $u\alpha$ a *conjugate* of u. It is clear that the conjugates of d and e are still isotropic. Our aim is to give a proof of the following result.

*Theorem*  Let $D_m$ be the set of linear combinations of the conjugates of the tensors d and e of order m, and let $U_m$ be the set of isotropic tensors of order m. Then $D_m = U_m$.

*Remark*  We have seen that $D_m \leq U_m$, so we have to show that $U_m \leq D_m$. Our proof uses induction on m, and our main tool is the conjugation action of $S_m$, which is said to have been first exploited by Schur, and developed by Weyl [4, page 96, Section 6]. We begin by deriving certain relations between the coordinates of an isotropic tensor.

*Lemma 1*  If u is isotropic, then

(a) $$\sum_{r=1}^{m} \delta(p,i_r)u(i_1, \ldots, i_{r-1}, q, i_{r+1}, \ldots, i_m) = \sum_{r=1}^{m} \delta(q,i_r)u(i_1, \ldots, i_{r-1}, p, i_{r+1}, \ldots, i_m)$$

(b) $$(n-1).u(i_1, \ldots, i_m) + \sum_{r=2}^{m} u(i_r, i_2, \ldots, i_{r-1}, i_1, i_{r+1}, \ldots i_m) = \sum_{r=2}^{m} \delta(i_1,i_r) \sum_{p=1}^{n} u(p, i_2, \ldots, i_{r-1}, p, i_{r+1}, \ldots i_m).$$

*Proof*  The equation (a) is trivial when $p = q$, so we suppose $p \neq q$. Define

$$T(h,k) = \begin{cases} \cos t & \text{when } h = k = p \text{ or } h = k = q, \\ \sin t & \text{when } h = p \text{ and } k = q, \\ -\sin t & \text{when } h = q \text{ and } k = p, \\ \delta(h,k) & \text{otherwise.} \end{cases}$$

Then $T(h,k)$ represents a rotation of the pq plane by an angle t, and is an orthogonal matrix with determinant 1. Therefore

$$\sum_{j_1=1}^{n} \ldots \sum_{j_m=1}^{n} T(i_1,j_1) \ldots T(i_m,j_m)u(j_1, \ldots ,j_m) = u(i_1, \ldots,i_m).$$

We differentiate this equation with respect to t, and then take $t = 0$. Note that, when $t = 0$,

$$T(h,k) = \delta(h,k), \quad \frac{dT}{dt}(h,k) = \delta(p,h)\delta(q,k) - \delta(p,k)\delta(q,h);$$

hence we get

$$\sum_{r=1}^{m} \sum_{j_1=1}^{n} \ldots \sum_{j_m=1}^{n} \delta(i_1,j_1) \ldots \delta(i_{r-1},j_{r-1}).(\delta(p,i_r)\delta(q,j_r)-$$

$$\delta(p,j_r)\delta(q,i_r)).\delta(i_{r+1},j_{r+1}) \ldots \delta(i_m,j_m)u(j_1,\ldots,j_m)=0.$$

Now $\delta$ has the substitution property [1, page 59, Section 2.021]

$$\sum_{j_r=1}^{n} \delta(i_r,j_r)u(j_1, \ldots ,j_m) = u(j_1, \ldots ,j_{r-1},i_r,j_{r+1}, \ldots ,j_m)$$

Using this, we can deduce (a) from the last equation.

In (a), take $i_1 = p$, and add the resulting relations for $p = 1,2, \ldots ,n$. This gives

$$n.u(q,i_2, \ldots ,i_m) + \sum_{r=2}^{m} u(i_r,i_2, \ldots ,i_{r-1},q,i_{r+1}, \ldots ,i_m)$$

$$= u(q,i_2, \ldots ,i_m) + \sum_{r=2}^{m} \delta(q,i_r)\sum_{p=1}^{n} u(p,i_2, \ldots ,i_{r-1},p,i_{r+1}, \ldots ,i_m).$$

Replacing q by $i_1$, we get (b).

*Definition* The equation (b) suggests the following notation and Lemma. Let $R_m$ be the rational group algebra of the symmetric group $S_m$ [3, page 42, Section 2.2]. The elements of $R_m$ are the expressions $\theta = \sum t_\alpha \alpha$, where the coefficients $t_\alpha$ are rational, and where $\alpha$ runs through $S_m$. If further, u is a tensor of order m, we define $u\theta = \sum t_\alpha(u\alpha)$, where $u\theta$ is given by the conjugation action. As usual, 1 denotes the identity permutation in $S_m$, and $(h,k)$ is the transposition which interchanges h and k, but fixes the other elements of $\underline{m}$. We write



$$\phi = (n-1).1 + (1,2) + (1,3) + \ldots + (1,m) \in R_m.$$

*Lemma 2* (a) If u is isotropic, then

$$u\phi = \sum_{r=2}^{m} \delta(i_1,i_r)\sum_{p=1}^{n} u(p,i_2, \ldots ,i_{r-1},p,i_r, \ldots ,i_m).$$

(b) $\{\theta \in R_m : U_m\theta \leq D_m\}$ is a (2-sided) ideal of $R_m$.

*Proof* (a) is a restatement of Lemma 1(b) in terms of the above definition. To prove (b), note that if $\alpha,\beta \in S_m$ then clearly $U_m\alpha = U_m$, $D_m\beta = D_m$. Hence $U_m\alpha\theta\beta \leq D_m\beta = D_m$ as required.

*Proof of the Theorem* We use induction on m. We interpret a tensor of order 0 as a *scalar* (a real number whose value does not depend on the choice of axes). This means that when $m=0$, then every tensor is a multiple of d, so the Theorem is trivial. When $m=n=1$, then a tensor is again the same as a scalar, and is a multiple of e, and so lies in $D_m$ as required. Next suppose $m=1$ and $n \geq 2$, and let u be isotropic. Taking $p \neq q$ and $(i) = (q)$ in Lemma 1(a), we get $u(p) = 0$ for all p, so the Theorem is again true. We have now proved the result when $m=1$ or 2, so we may suppose $m > 2$, and assume that the Theorem holds for orders less than m.

The contracted tensor [1, page 87]

$$\sum_{p=1}^{n} u(p,i_2, \ldots ,i_{r-1},p,i_{r+1}, \ldots ,i_m)$$

is clearly isotropic of order $m-2$, so it lies in $D_{m-2}$ by the inductive hypothesis. Hence Lemma 2(a) implies that $u\phi \in D_m$ whenever u is isotropic, and therefore

(1) $$U_m\phi \le D_m.$$

The following result will be proved later.

*Lemma 3* Suppose $1 \le m < n$, and let X be the ideal of $R_m$ generated by $\phi$. Then $1 \in X$.

It follows from (1) and Lemma 2(b) that $U_mX \le D_m$. Assuming Lemma 3, we deduce that $U_m \le U_mX \le D_m$. This proves the Theorem when $m < n$.

*Remark* For each subset $\{i_1, i_2, \ldots, i_r\}$ of $\underline{m}$, define

$$\pi(i_1, i_2, \ldots, i_r) = \sum e_\alpha \alpha$$

where $\alpha$ runs through the permutations of $\{i_1, i_2, \ldots, i_r\}$ and where

$$e_\alpha = \begin{cases} 1 & \text{if } \alpha \text{ is even,} \\ -1 & \text{if } \alpha \text{ is odd.} \end{cases}$$

It can be shown that if $m = n$, then $\phi.\pi(1,2, \ldots, m) = 0$, which implies that the conclusion of Lemma 3 no longer holds. Indeed, the tensor $\varepsilon$ appears in $U_m$ for the first time in this case, so a different argument is needed.

Returning to the proof of the Theorem, suppose that $m = n > 2$, and that u is isotropic. Taking $p = 2$, $q = 1$, and $(i) = (2,2,3,4, \ldots, m)$ in Lemma 1(a), we get

$$u(1,2, \ldots, m) = -u(2,1,3,4, \ldots, m).$$

Similarly, one can show that, if the polyindex $(i) \in \underline{m}^{\underline{m}}$ is a permutation, then interchanging any 2 indices alters the sign of u(i). Since $S_m$ is generated by transpositions [2, page 136, Theorem 21], it follows that if (i) is a permutation, and



if $u_0 = u(1,2, \ldots, m)$, then $u(i) = u_0\varepsilon(i)$. Writing $v(i) = u(i) - u_0\,\varepsilon(i)$, we deduce that v is an isotropic tensor with

(2) $$v(i) = 0 \text{ when } (i) \text{ is a permutation.}$$

But if (i) is not a permutation, then it must have a repeated entry. Now if $i_1 = i_2$ and $\theta = 1 - (1,2) \in R_m$, and if w is any tensor, then

$$(w\theta)(i) = 0.$$

We note also that if $\mu$ is the sum of the even permutations of $\underline{m}$, then $\mu\theta = \pi(1,2, \ldots, m)$ [2, page 137, Proposition 26], whence the last equation implies that $(w.\pi(1,2, \ldots, m))(i) = 0$ when $i_1 = i_2$. Similarly, one can see that if (i) is a polyindex with a repeated entry, and if w is any tensor of order m, then $(w.\pi(1,2, \ldots, m))(i) = 0$. Combining this with (2), we conclude that

(3) $$(v.\pi(1,2, \ldots, m))(i) = 0 \text{ for all } (i) \in \underline{m}^{\underline{m}}.$$

The following result will be proved later.

*Lemma 4* Suppose $m = n \ge 1$, and let Y be the ideal of $R_m$ generated by $\pi(1,2, \ldots, M)$ and $\phi$. Then $1 \in Y$.

Copying the proof of Lemma 2(b), we see that if $V_m$ is the set of isotropic tensors v which satisfy (3), then $\{\theta \in R_m : V_m\theta \le D_m\}$ is an ideal of $R_m$. Using (1) and (3), we deduce that $V_mY \le D_m$, so Lemma 4 implies that

$$v = v1 \in V_mY \le D_m.$$

Hence $u = u_0\varepsilon + v \in D_m$ as required.

Finally, suppose $m > n$. Then there must be a repeated index among $i_1, i_2, \ldots, i_{n+1}$, and it follows as before that $u.\pi(1,2, \ldots, n+1) = 0$. In the same way as in the previous case, the Theorem is a consequence of the following result, which will be proved later.

*Lemma 5* Suppose $m > n \geq 1$, and let $Z$ be the ideal of $R_m$ generated by $\pi(1,2, \ldots ,n+1)$ and $\phi$. Then $1 \in Z$.

It now remains to prove the Lemmata 3, 4 and 5. This will be done with the help of the following calculations. We write

$$\underline{\theta} = n.1 + (1,2) + (1,3) + \ldots + (1,r) \in R_m.$$

*Lemma 6* (a) $\pi(1,r+1,r+2, \ldots ,m) = (1,r).\pi(r,r+1, \ldots,m).(1,r)$

(b) $\underline{\theta}.\pi(r+1, r+2, \ldots ,m) = \phi.\pi(r+1,r+2, \ldots ,m) + \pi(1,r+1,r+2, \ldots ,m).$

*Proof* (a) follows from the rule for conjugating permutations [2, pages 129-130]. To prove (b), let $G$ and $H$ be the symmetric groups of permutations of the sets $\{1,r+1,r+2, \ldots ,m\}$ and $\{r+1,r+2, \ldots ,m\}$ respectively. Then $H$ is a subgroup of $G$ of index $|G:H| = (m-r+1)!/(m-r)! = m-r+1$. If $i \neq j$ then $(1,i)^{-1}(1,j) = (1,i,j) \notin H$ and therefore the cosets $(1,i)H$ and $(1,j)H$ are distinct [2, page 33, Proposition 5]. It follows [2, page 34] that $\{1,(1,r+1),(1,r+2), \ldots ,(1,m)\}$ is a set of representatives (or *transversal*) for the cosets of $H$ in $G$, and that

$$G = H \cup (1,r+1)H \cup (1,r+2)H \cup \ldots \cup (1,m)H.$$

Noting that multiplication by a transposition $(1,i)$ changes even to odd permutations and *vice-versa*, we deduce that

$$\begin{aligned}\pi(1,r+1,r+2, \ldots ,m) &= (1 - (1,r+1) - (1,r+2) - \ldots - (1,m)).\pi(r+1,r+2,\ldots ,m)\\ &= (\theta - \phi).\pi(r+1,r+2, \ldots ,m),\end{aligned}$$

which is equivalent to the required relation.



*Proof of Lemma 3* If $m = 1 < n$, then $\phi = (n-1),1$, whence $1 \in X$. We may therefore suppose $m > 1$, and use induction on $m$. In particular, the inductive hypothesis allows us to assume that if $1 \leq r < m$, then 1 is in the ideal generated by $\theta$, so there are permutations $\alpha_i$, $\beta_i$ of $\underline{r}$ such that

$$(4) \qquad 1 = \sum \alpha_i \theta \beta_i.$$

We shall prove by induction on $r$ that

$$(5) \qquad \pi(r+1,r+2, \ldots ,m) \in X \qquad (0 \leq r < m).$$

We note first that if $1 < i \leq m$, then $(1,i).\pi(1,2, \ldots ,m) = -\pi(1,2, \ldots ,m)$ and hence $\phi.\pi(1,2, \ldots ,m) = (n-m).\pi(1,2, \ldots,m)$. Since $n > m$, it follows that (5) holds when $r = 0$. We may therefore suppose $r > 0$, and assume that $\pi(r,r+1, \ldots ,m) \in X$. Then $\pi(1,r+1,r+2, \ldots ,m) \in X$ by Lemma 6(a). Using Lemma 6(b) we deduce that

$$\theta.\pi(r+1,r+2, \ldots ,m) \in X.$$

Now the permutations $\beta_i$ in (4) are disjoint from $\pi(r+1,r+2, \ldots ,m)$ and so commute with it. Hence we can multiply (4) by $\pi(r+1,r+2, \ldots ,m)$ to get

$$\pi(r+1,r+2, \ldots ,m) = \sum_i \alpha_i \theta.\pi(r+1,r+2, \ldots ,m)\beta_i \in X.$$

This proves (5). Taking $r = m-1$, we conclude that $1 = \pi(m) \in X$, as required.

*Proof of Lemma 4* We shall prove by induction on $r$ that

$$(6) \qquad \pi(r+1,r+2, \ldots ,m) \in Y \qquad (0 \leq r < m).$$

By the definition of $Y$, (6) holds when $r = 0$, so we may suppose $r > 0$, and assume that $\pi(r,r+1, \ldots ,m) \in Y$. Using Lemma 6, as above, we deduce that $\theta.\pi(r+1,r+2, \ldots ,m) \in Y$. Since $m = n > r$, it follows from Lemma 3 that 1 is in the ideal generated by $\theta$. As before, this enables us to prove (6), and we get the result by taking $r = m-1$.

*Proof of Lemma 5* If $m = 2$, then $n = 1$ and $\phi = (1,2)$, so $1 = \phi.(1,2) \in Z$. We may therefore suppose $m > 2$, and use induction on $m$. In particular, we may assume that if $n+1 < r < m$ then 1 is in the ideal generated by $\pi(1,2, \ldots ,n+2)$ and $\phi$. Since

$$\pi(1,2, \ldots ,n+2)$$

$$= (1 - (1,n+2) - (2,n+2) - \ldots -(n+1,n+2)).\pi(1,2, \ldots ,n+1)$$

it follows that there are permutations $\alpha_i$, $\beta_i$, $\lambda_j$, $\mu_j$, such that

$$(7) \qquad 1 = \sum_i \alpha_i \theta \beta_i + \sum_j \lambda_j \pi(1,2, \ldots ,n+1)\mu_j,$$

provided $n+1 < r < m$. If $r = n+1$, then the same result follows from Lemma 4. Moreover, if $r < n+1$, then Lemma 3 implies that 1 is in the ideal generated by $\theta$, so we again get the equation (7), but now with $\lambda_j = \mu_j = 0$. Thus (7) holds whenever $r < m$.

We shall prove by induction on $r$ that

$$(8) \qquad \pi(r+1,r+2, \ldots ,m) \in Z \quad (m-n-1 \leq r < m).$$

If $\gamma$ is in the m-cycle $(1,2, \ldots ,m) \in S_m$, then

$$\pi(m-n,m-n+1, \ldots ,m) = \gamma^{-m+n+1}\pi(1,2, \ldots ,m)\gamma^{m-n-1} \in Z,$$

so (8) is true when $r = m-n-1$. We may therefore suppose that $r \geq m-n$, and assume that $\pi(r,r+1, \ldots ,m) \in Z$. Using Lemma 6, we deduce that $\theta.\pi(r+1,r+2, \ldots ,m) \in Z$. As before, we can multiply (7) by $\pi(r+1,r+2, \ldots ,m)$ to get

$$\pi(r+1,r+2, \ldots ,m) = \sum_i \alpha_i \pi(r+1,r+2, \ldots ,m)\beta_i$$

$$+ \sum_j \lambda_j \pi(1,2, \ldots ,n+1)\pi(r+1,r+2, \ldots ,m)\mu_j \in Z.$$

This proves (8). Taking $r = m-1$, we obtain the required result.

*Remark* It can be shown that if $1 \leq m < n$, then 1 is in the right or left ideal generated by $\phi$. We do not know whether



Y or Z can be replaced by a one-sided ideal in Lemma 4 or 5.

We mentioned in an earlier remark that $\pi(1,2, \ldots ,m)$ is needed in Lemma 4. It can also be shown that if either $m = 2$ and $n = 0$, or $m = 4$ and $n = 1$, or $m = 6$ and $n = 2$, then 1 is not in the ideal of $R_m$ generated by $\phi$. However, we do not know for which values of $m$ and $n$, $\pi(1,2, \ldots ,n+1)$ is needed in Lemma 5.

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# GROUPS AND TREES

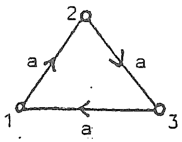
*John McDermott*

For some years now, the M.Sc. Algebra course in U.C.G. has included a unit on group presentations based mainly on the first half of Johnson's book [4]. One of the attractions of the book is its emphasis on computational aspects of the subject such as coset enumeration. On the occasions that I taught this unit, I have experimented with the use of graph-theoretical ideas in the presentation of supplementary material and also to provide an alternative approach to some of the topics in the text. The graph theory involved uses little more than basic concepts (in particular, a course on graphs is not a pre-requisite); but the intuitive "geometric" framework it provides is, I believe, helpful to the student. I made some comments along these lines in a talk at the DIAS Symposium in December 1982. In this note, which is based on that talk, I outline some of the graph-theoretical approaches to group presentation.

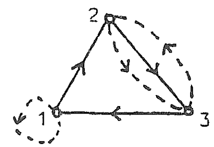
A *graph* is usually defined to be a (non empty) set V of *vertices* (points), some pairs of which are joined by *edges*. In the context of group presentations we should, strictly, talk about *directed multigraphs*: that is, an edge may be directed from one vertex to the other, and there may be several edges between two given vertices. Moreover, the graph may be *coloured*: its vertices and/or its edges may have colours (labels) attached; these are usually elements of G, some group related to the graph. In particular, we adopt the convention that an edge from x to y labelled g is implicitly an edge from y to x coloured by $g^{-1} \in G$.

The classical examples of graphs related to group presentations are *Cayley diagrams* and *Schreier diagrams*. Let G be a group generated by a,b, ... ; the corresponding Cayley diagram is the graph having the elements of G as vertices and for each generator, such as a, an edge from x to y coloured with a if y = xa. If H is a subgroup of G, the corresponding Schreier diagram has the cosets of H in G as its vertices and, for each generator a, an edge labelled a from Hx to Hy if Hxa = Hy. There are examples in Fig. 1.





(i)



(ii)

FIGURE 1

In (i) we have the Cayley diagram for $G = \langle a | a^3 = 1 \rangle$, the cyclic group of order 3; for convenience, we use i to denote the vertex $a^{i-1}$, and an arrow on each edge indicating its direction.

In (ii) we have the Schreier diagram of G with respect to H, where $G = \langle a,b | a^3 = b^2 = (ab)^2 = 1 \rangle$, the non-abelian group Sym (3) of order 6, and $H = \langle b \rangle$, a subgroup of order 2. The solid edges are each coloured a in the direction of the arrow, the dotted edges are coloured b (in either direction) and the vertex $Ha^{i-1}$ is denoted by i.

A Schreier diagram is particularly useful in the study of a group presentation. We can read off from it not only the index of H in G - which is just the number of vertices in the diagram - but also the images of the generators of G in the permutation representation of G on the cosets of H. For example, we can see in (ii) that

$$a \longmapsto (123) \quad \text{and} \quad b \longmapsto (23)$$

One may view the construction of a Schreier diagram as a pictorial implementation of the technique of *coset enumeration*; and with a little elaboration it may also be used to describe the process of finding a presentation for H in terms of the given presentation for G. There is a nice informal description of some of these ideas in the first section of Chapter VIII of the book [1] by Bollobas.

Now we need a couple of definitions. A *path* from x to y in a given graph is what intuition suggests, essentially a sequence $x_0, x_1, \ldots, x_k$ of vertices such that $x_0 = x$, $x_k = y$ and each pair $x_{i-1}$, $x_i$ is joined by an edge. If some of the edges are directed, we do not insist that the direction is from $x_{i-1}$ to $x_i$; moreover, if there are several edges between $x_{i-1}$ and $x_i$ it is necessary to indicate which edge is intended in the path. A *tree* is a graph in which, given any distinct vertices x and y, there is a unique path from x to y; equivalently, a tree is *connected* (there is a path between any two vertices) *and contains no cycle* (path with $x_k = x_0$, $k > 0$, not using any edge twice). It can be shown, by successive deletion of edges from cycles, that any connected graph contains a *spanning tree*, that is a tree using all the original vertices. For example, the graph in Fig. 1(ii) is obviously connected; it contains cycles such as 1231, and two of its spanning trees are shown in Fig. 2.


FIGURE 2

In the last fifteen years, there has been a new and striking use of graphs to illustrate the theory of group presentations, in particular the basic constructions such as free groups and free products. This is the Bass-Serre theory of *groups acting on trees*, and the basic reference is Serre's book [6]. The starting point is the following easy observat-



ion. The Cayley diagram T(F) of a free group F with respect to a set of free generators is a tree: the "generating" property says that T(F) is connected, and the "free" property says that it has no cycles. Moreover, F acts by left multiplication as a group of automorphisms of T(F) and the action is free (no vertex or edge is fixed by any non-identity element of F). This property actually characterises free groups:

THEOREM: A group F is free if and only if it acts freely on some tree.

Remark: The proof of the "if" part is non-trivial, but it is possible to extract a reasonably simple account from Sections 2 and 3 of the first chapter in [6]. The proof actually produces a set of free generators for F. A key technical point is that given a spanning tree $T_0$ of the natural quotient graph T/G, there is a subtree of T which projects isomorphically onto $T_0$.

An immediate pay-off is *Schreier's subgroup theorem*:

COROLLARY: If F is a subgroup of a free group $F_0$ then F is free.

Proof: The theorem ensures that $F_0$ acts freely on some tree T. Since F is a subgroup of $F_0$, it acts freely on the same T, and hence it is free.

Remark: In the situation of the corollary, consider F as acting of $T(F_0)$, and choose a spanning tree $T_0$ in the Schreier diagram of $F_0$ with respect to F. Then the free generators for F produced by the theorem are in one to one correspondence with those edges of the diagram that do not belong to $T_0$. Using this fact, it is easy to establish the *Schreier index formula* for the rank (number of free generators) of F when the index $|F_0 : F|$ is finite. Moreover, there is a link here with the classical approach to these matters. The spanning tree $T_0$ corresponds to a certain *Schreier transversal* for $F_0$ in F: choose as coset representative for a given vertex i the product of the colours on the unique path in $T_0$ from 1 to i. For example, recall the groups G and H used to illustrate the idea of a Schreier diagram in Fig. 1(ii). If we interpret G as $F_0/N$, where $F_0$ is the free group on a and b and N is the normal subgroup of $F_0$ generated by $a^3$, $b^2$ and $(ab)^2$, we may consider Fig. 1(ii) as the Schreier diagram of $F_0$ with respect to F, where F is the pre-image of H in $F_0$. If we choose $T_0$ as in Fig. 2(i) then the corresponding Schreier transversal is $\{1, a^{-1}, a^{-1}b\}$; and since there are four edges in Fig. 1(ii) that are not in $T_0$ we have rank F = 4.

What else can be studied in the graph-theoretical framework? By way of illustration, consider the following result.

PROPOSITION: A group G is a free product if and only if there is a tree on which G acts (i) regularly on the edges but (ii) not transitively on the vertices.

Proof: Assume that G acts with properties (i) and (ii) on the tree T, and let e = xy be a particular edge in T. Property (i) says that, given any edge f in T there is a unique element of G which moves e to f; we label f by the corresponding $g \in G$, so that e is labelled by 1. It follows that every vertex is in the same orbit as (that is, can be moved to) at least one of x or y; but in view of (ii) there are then exactly two vertex orbits, and the end vertices of any edge are in different orbits. Hence, the edges $\neq$ e that meet x (respectively y) are labelled by the non-identity elements of $A = G_x$, (respectively $B = G$



the stabiliser of x (respectively y). We note that $A \cap B = 1$ by (i). A routine, if tedious, argument shows that each edge $\neq$ e is labelled by an alternating product of non-trivial elements from A and B, thus identifying G with the free product A*B.

Conversely, given G = A*B, construct T as follows: the vertices of T are the cosets of A and B in G, and Ag is joined to Bh if and only if $Ag \cap Bh \neq \phi$. It is clear that G acts on T by right multiplication and that there are two orbits of vertices, so (ii) holds. The standard properties of the free product ensure that T is a tree and that (i) holds also.

A familiar fact follows readily:

COROLLARY: Let H be a subgroup of the free product G = A*B. If no conjugate of H meets A or B non-trivially then H is free.

Proof: Let G act on a tree T as in the proposition. For any vertex z of T the stabiliser $G_z$ is conjugate in G to A or to B according as z is in the orbit of A-cosets or the orbit of B-cosets. Thus the hypothesis of the corollary ensures that $H \cap G_z = 1$; in other words, H acts freely on the vertices of T. Since H, being a subgroup of G, also acts freely on the edges of T, we deduce from the earlier theorem that H is free.

Remarks: (a) It is possible to produce similar 'special case' treatments for *free products with amalgamation* and for *HNN groups*; whether this is worth doing depends on, among other things, the amount of time available for the course. It might be argued that if there is time for several special cases then one should treat the general structure theorem of I.5 [6]. However, I feel that students should meet the various constructions separately in a first treatment; the general theorem can follow if there is time - and a desire - for it.

(b) It may be worth mentioning the somewhat surprising fact that the theory of groups acting on (infinite trees is significant in the study of finite groups; see, for example, Goldschmidt's article [3].

(c) For a more topological account of the Bass-Serre Theory see Cohen's notes [2] or the Scott-Wall article [5].

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# AN INTUITIVE PROOF
# OF BROUWER'S FIXED POINT THEOREM IN $R^2$

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Fixed point theorems play a major role in general equilibrium theory. Brouwer's theorem is the most basic of these; it states that any continuous function mapping a closed bounded convex set into itself must contain at least one fixed point (i.e., a point that is its own image).

Elementary discussions invariably give an intuitive proof of the theorem for functions of a single variable, as illustrated in Fig. 1. In $R^1$ a set is convex if and only if it is an interval; thus a continuous mapping of the closed boundary interval $[x_0, x_1]$ into itself can be represented by a curve f.

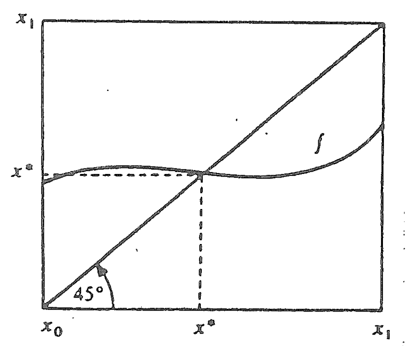


FIGURE 1

Since f connects the left-hand side of the rectangle to the right-hand side of the rectangle, it is intuitively obvious that f must intersect the diagonal of the rectangle at least

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once, and at this point $f(x^*) = x^*$. A bit more formally, if $f(x_0) \neq x_0$ and $f(x_1) \neq x_1$, then $\phi(x_0) = f(x_0) - x_0 > 0$ and $\phi(x_1) = f(x_1) - x_1 < 0$. Since $\phi$ is continuous on $[x_0, x_1]$, the intermediate value theorem implies that $\phi$ must assume the val zero somewhere on the open interval $(x_0, x_1)$, which proves the theorem.

An intermediate- or advanced-level student should be a bi street-wise and skeptical of the validity of demonstrations based on two-dimensional diagrams. The purpose of this note is to demonstrate that the intuitive graphic proof generalize to three dimensions (i.e., to functions on $R^2$) and can be mad rigorous at that level.

To begin, let W be any closed bounded (i.e., compact) co vex set in $R^2$ and let f be any continuous function mapping W into itself. Since W is bounded it can be contained in a re angle as shown in Fig. 2. We may now extend f to the closed

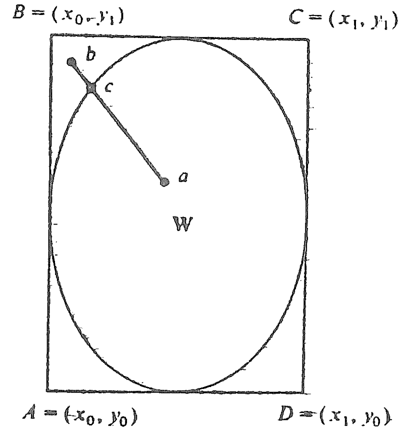


FIGURE 2

rectangle ABCD as follows. Choose an arbitrary interior poi a in W and for each point b in the rectangle but not in W, define f(b) to be the image of the point c at which the line through a and b intersects the boundary of W. The extended



napping is continuous and maps the closed rectangle ABCD into W. For simplicity we denote the extended mapping by f and represent f by $f(x,y) = (x',y')$ where

$$x' = g(x,y)$$

$$y' = h(x,y) \qquad (1)$$

with $x, x' \in [x_0, x_1]$; $y, y' \in [y_0, y_1]$; and with g and h continuous.

Fig. 3 gives a three-dimensional representation with W and the rectangle ABCD in the horizontal coordinate plane. Represented above the two-dimensional rectangle ABCD is the three-dimensional box EIJFGKLH. The sides EI, FJ, GK and HL as well as EH, IL, JK and FG all correspond to the interval $[x_0, x_1]$. Similarly, the sides EF, IJ, LK and HG all correspond to the interval $[y_0, y_1]$. The graph of g is given by the surface MNOP which is restricted to the closed three-dimensional box since x' is restricted to $[x_0, x_1]$.

Now consider the projection mappings $p_x$, $p_y$ defined by

$$x = p_x(x,y)$$

$$y = p_y(x,y) \qquad (2)$$

The graph of $p_x$ in Fig. 3 is the diagonal plane EFKL and the intersection of g and $p_x$ is the manifold RQ which projects into the horizontal coordinate plane as TS. Since neither the surface MNOP nor the diagonal plane EFKL have any rips in them, it is intuitively obvious that the intersection of g and $p_x$ must connect the face and back of the three-dimensional box and that the projection TS connects opposite sides of ABCD. Further, TS represents the points (x,y) in ABCD for which $x' = x$. Similarly, the intersection of h and $p_y$ projected to the coordinate plane will connect the left and right sides of ABCD as UV does in Fig. 3. This projection represents the points (x,y) in ABCD for which $y' = y$. Again, intuition tells us that UV must intersect TS (at least once) and any intersection of UV and TS

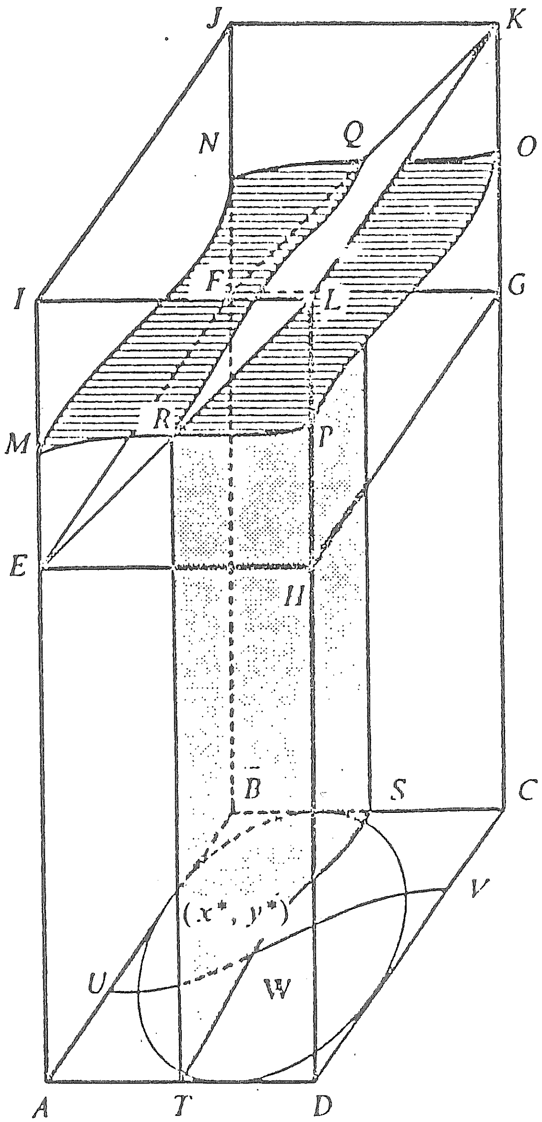


FIGURE 3



is a fixed point of f.

To make this demonstration rigorous, it is necessary to prove that TS (or UV) actually connects opposite sides of ABCD. As a first step we show that if Brouwer's theorem holds for functions which are "very close" to f, then it must hold for f itself. Let $\|q-r\|$ denote the usual Euclidean distance between two points q,r in $R^2$. For any given $\varepsilon > 0$, suppose that there exists a continuous function $f^*$:ABCD → ABCD such that $\|f^*(x,y)-f(x,y)\| \leq \varepsilon$ for all $(x,y) \in$ ABCD, and such that $f^*$ has a fixed point in ABCD. We claim that this property implies that f has a fixed point in ABCD. Applying the property, we can assume that for each n = 1,2,3,... there exists a continuous function $f_n$:ABCD → ABCD such that

$$\|f_n(x,y)-f(x,y)\| \leq \frac{1}{n}$$

for all $(x,y) \in$ ABCD, and there is a point $Z_n \in$ ABCD such that $f_n(Z_n) = Z_n$. The compactness of ABCD implies that the sequence $\{Z_n\}$ has a limit point, $Z^*$. We invite the reader to show that $Z^*$ is a fixed point of f.

It is thus sufficient to replace f by another function which closely approximates f, then prove the Brouwer theorem for the replacement function. The Weierstrass approximation theorem (a generalized version is proven in [4, §36]; for the specific $R^2$ case see [2, p. 187, problem 2]) yields, for a given $\varepsilon > 0$, a function $\hat{f} = (\hat{f}_1, \hat{f}_2)$:ABCD → $R^2$ such that $\|f(x,y)-\hat{f}(x,y)\| \leq \varepsilon$ for all $(x,y) \in$ ABCD, with $\hat{f}_1$ and $\hat{f}_2$ polynomials in x and y. However $\hat{f}$ may give values lying at a distance $\varepsilon$ outside of ABCD, so we must shrink its range slightly. To do this, replace $\hat{f}_1(x,y)$ by

$$\frac{x_0 + x_1}{2} + (x_1 - x_0)\left(\frac{\hat{f}_1(x,y) - \frac{x_0 + x_1}{2}}{x_1 - x_0 + 4\varepsilon}\right)$$

and replace $\hat{f}_2$ by a similar expression. A short calculation shows that these new functions (which for simplicity we again call $\hat{f}_1$ and $\hat{f}_2$ ) approximate f and give us $\hat{f} = (\hat{f}_1, \hat{f}_2)$:ABCD → interior of ABCD.

Now define $\hat{g}$ on ABCD by:

$$\hat{g}(x,y) = \hat{f}_1(x,y) - x$$

Then

$$\hat{g} > 0 \text{ on AB and } \hat{g} < 0 \text{ on CD} \qquad (3)$$

We must modify $\hat{g}$ still further so that its partial derivatives satisfy certain conditions, while retaining property (3). First, on AD, $y = y_0$ is constant so on AD $\hat{g}(x,y) = \hat{g}(x,y_0)$ is just a polynomial in x. By altering $\hat{g}(x,y)$ slightly if necessary we can ensure that $\hat{g}(x,y_0)$ has no repeated factors. There are then no points on AD where $\hat{g}(x,y)$ and $\partial\hat{g}(x,y)/\partial x$ vanish simultaneously. A further slight perturbation of $\hat{g}$ will ensure that at least one partial derivative of $\hat{g}$ is non-zero at each point in ABCD where $\hat{g}(x,y) = 0$. This assertion follows from Sard's theorem ([3], [6, Chapter 13, §14]; or for a proof of a special case of this theorem which can easily be adapted to the present situation, see [1, p. 35]).

We can now proceed directly. By the implicit function theorem (see any advanced calculus book) the above condition on $\partial\hat{g}/\partial x$ and $\partial\hat{g}/\partial y$ in ABCD guarantees that $\hat{g}^{-1}(0)$ is a simple one-dimensional curve in a neighbourhood of each point on $\hat{g}^{-1}(0)$. Consequently $\hat{g}^{-1}(0)$ is a collection of simple curves, no two of which intersect. Wherever one of these curves intersects AD, our earlier condition on $\partial\hat{g}/\partial x$ guarantees that the curve is not tangent to AD. By (3) and the $\partial\hat{g}/\partial x$ condition, the number of points in AD ∩ $\hat{g}^{-1}(0)$ is odd, since at each such point $\hat{g}(x,y_0)$ changes sign. Curves which originate and terminate on AD account for an even number of these points so there must be a curve that has only one endpoint on AD. The other end of this curve cannot be on AB or CD by (3), so it must join AD and BC and we are free to label the endpoints T and S.



To complete the proof, we define $\hat{h}(x,y) = \hat{f}_2(x,y) - y$. Clearly $\hat{h}$ is continuous. Since $\hat{h}(S) \le 0 \le \hat{h}(T)$, the intermediate value theorem assures us of the existence of at least one point $(x^*,y^*)$ on TS such that $\hat{h}(x^*,y^*) = 0$ or (equivalently) $\hat{f}_2(x^*,y^*) = y^*$. Since all points $(x,y)$ on TS satisfy $\hat{g}(x,y) = x$, we have $\hat{g}(x^*,y^*) = x^*$. Thus there exists at least one point $(x^*,y^*)$ in ABCD such that $(x^*,y^*) = \hat{f}(x^*,y^*)$. Since $\hat{f}$:ABCD → W we must have $(x^*,y^*) \in W$. Thus $(x^*,y^*)$ is a fixed point of $\hat{f}$.

It is well known that in the class of compact sets, the fixed point property is not restricted only to convex sets [5, p. 9]. It can be shown that if a set has the fixed point property, then any set to which it is homeomorphic also has the fixed point property [5, p. 9]. This theorem can be used to prove that various plane sets with amoeboid shapes have the fixed point property. Our proof given above shows that *any bounded set S in $R^2$ having an interior point x such that each ray from x has only one intersecting point with the boundary of S has the fixed point property.*

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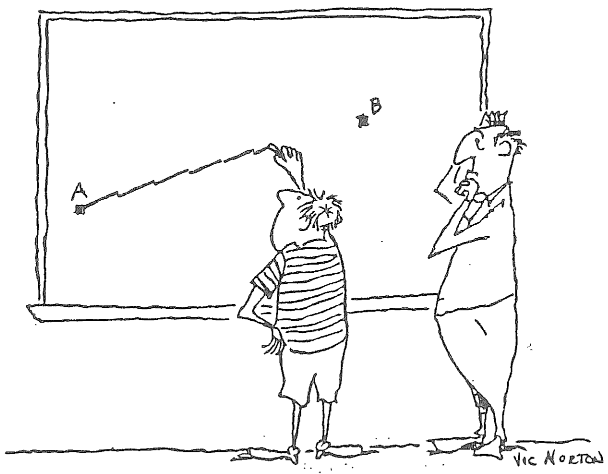
**Cartoon without caption:**
**The computer-age generation**

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# THE ANALYTICAL REFORM OF IRISH MATHEMATICS

## 1800 - 1831

*N.D. McMillan*

### The Origin of the Dublin Mathematical School

The mathematical tradition established by the Dublin Philosophical Society of William Molyneux (Fig. 1) had a major influence on the character of mathematics in Ireland [1]. The convergence of interests at the University of Dublin on specific aspects of mathematics, e.g. the theory of equations, optics, potential theory and variational principles [2], and the strong Irish tradition in statistics [3] had their origins in the interests and contributions of the members of the society.

| | |
|---|---|
| STATISTICS | W. Petty, *Political Arithmetick* (London, 1690).<br>F. Robartes, *An Arithmetical Paradox Concerning the Chances of Lotteries*, Phil. Mag. XVII (1693) pp.677-84. |
| GEOMETRY | St. George Ashe, *A New and Easy Way of Demonstrating Some Propositions in Euclid*, Phil. Mag. XIV (1684), pp. 672-6. |
| OPTICS | W. Molyneux, *Solution of a Dioptric Problem*, Bibliotheque Universelle et Historique, III (1686). |
| ENGINEERING MATHEMATICS | W. Molyneux, *A Demonstration of an Error Committed by Common Surveyors .....*, Phil. Mag., XIX (1677) pp. 625-31. |
| ASTRONOMY | W. Molyneux, *Concerning the Parallax of Fixed Stars*, Phil. Mag. VXII (1693) pp. 844-9.<br>J. Walley, *Ptolemy's Quadripartite*, (Dublin, 1701). |
| ACOUSTICS | N. Marsh, *An Introductory Essay to the Doctrine of Sounds*, Phil. Mag. VIX (1684) pp. 472-88. |

FIGURE 1: Mathematical Interest of the Dublin Philosophical Society Illustrated by Selection of Works.

The 18th century began on a high mathematical note at the University of Dublin with the first fundamental Critique of Newton by George Berkeley. He began tentatively his criticism of Newton's cosmology with his *Essays Towards a New Theory of Vision* (1709) and ended forcefully in his apology for theology, *The Analyst* (1734) which pointed out the unsatisfactory state of the underlying logical base of the Newtonian calculus [4]. Newton had disguised the inadequate logical base of his calculus by juggling away higher order terms which experiment had shown to be extraneous. "The good Bishop" took issue with Newton (and Leibniz) demanding proof of the truth of the calculus as a representative of material motion, rather than a utilitarian defence of the methodology whose inadequacies the founders of the calculus camouflaged by a good deal of mysticism. Berkeley was not the only man in Britain to criticize Newton [5] and indeed in Dublin there were anti-Newtonians during this period [6].

In the 18th century there was apparently little original work in mathematics at the University of Dublin except for the work of Hugh Hamilton (1729-1783) and Richard Murray ( ? - 1799) which was to lay the foundation of the Dublin Mathematical School. Hamilton wrote the elegant work *De sectionibus, conicis, tractatus geometricus* (1758) which earned the following accolade from Leonhard Euler:

> "There are three perfect mathematical works: these are by Archimedes, Newton and Hamilton."

In addition he wrote *The Analysis of the Infinities* (date uncertain) and *Lectures on Natural Philosophy* (1766). Murray was professor of mathematics from 1764 (two years after the chair was established) to 1795 and devoted his energies to the improvement of mathematics at the University of Dublin. From his patient teaching arose a broader mathematical base in the University and, in particular, during this period mathematics became the most important single subject in the Fellowship examinations which were then the only method of entry into this academic world [7].



Someone who was strongly influenced by Murray was Matthew Young (1750-1800) whose books such as *An Enquiry into the Principal Phenomena Of Sounds and Musical Strings* (1784) [8] contained some original researches. However, Young's most significant contribution to Dublin mathematical research was his role in the establishment of the Royal Irish Academy [9]. The Academy was dedicated to the development of new knowledge in contrast to the Dublin Philosophical Society's commitment to dissemination and diffusion of technical knowledge. William Hales also wrote several books on mathematical subjects [10] and his *Analysis aequationium* (1786) was complimented in a letter from Lagrange. Hales provided a thoroughgoing attempt to defend Newton's fluxian notion in his *Analysis Fluxionum* which he wrote after he had retired.

Young died leaving unfinished a work of great scholarship on Newton's *Method of Prime and Ultimate Ratios, Illustrated by a Comment of the "Principia"* [11]. The suppression of this work was unfortunate for mathematics in Dublin. Apparently it resulted from some feared heterodox doctrinal deviation by the pious polymath Young [12]. As for mathematical deviation, Rev. William Jones, the author of *Essays on the First Principles of Natural Philosophy*, claimed that in Dublin there were mathematicians who kept guard for the system of attractions "more severely that Newton himself did and would not suffer a heretic to land on their wast" [13].

The days of orthodox Newtonianism had, by the time of Jones' comment, already passed in Dublin and the old University mathematical establishment was to be superseded by a new generation of analytical reformers in the next period. The arrival of John Brinkley from Cambridge as the new Andrew's Professor of Astronomy in 1790 was the turning point. The growth of his influence in Dublin progressively subverted the old Newtonian tradition.

## The Gentlemen of Science, Rev. John Brinkley and Rev. Bartholomew Lloyd and the Reform Movement

The Dublin School of Mathematics [14] was the creation of two men, Bartholomew Lloyd and John Brinkley. In 1792 Brinkley became the first Astronomer Royal of Ireland [15] and, while he waited patiently for more than a decade for the arrival of the great Ramsden Circle for Dunsink, he prepared himself with magnificent thoroughness for his later parallax observations with the Circle [16]. In 1800 he was already demonstrating a great awareness of the work of Lagrange and Laplace. It is evident that at the University of Dublin, Brinkley was the catalyst for the reform of mathematics there since he was the most senior University reformer in the period of great political reaction following the Rebellion [17]. He was isolated from University life because of his position at Dunsink and it was perhaps because of this that he was able to take the lead in introducing a knowledge of continental mathematics and physics into Dublin without raising a hornets' nest of opposition in the University. He was the first Dublin professor to use the analytic notation [18] early in the new century and this is of great significance in understanding the roots of the advanced analysis in Dublin.

The period 1798-1830 was an age of Tory hegemony at the University of Dublin, but one in which there must have been a working relationship existing between, on the one hand, the Tory Provosts and Tory majority on the Board and, on the other hand, the reformers. The reformers' roots were the old Whig establishment of the University. Lloyd, who was a member of a third generation of mathematicians at the University from this reforming tradition (Fig. 2), was evidently a trusted radical in that in 1813, when still a Junior Fellow and comparatively a young man, he was appointed to the Chair of Mathematics in an unprecedented promotion. As a reformer, Lloyd was in tune with the needs of his age and he has been credited with the single-handed reform of the University's mathematical curriculum [19], but this is far from the whole truth. The working



| | | | | |
|---|---|---|---|---|
| W. Molyneux (1656-1698) | St. George Ashe (1657-1718) | | G. Berkeley (1685-1753) | W. Petty (1625-1637) |
| H. Hamilton (1729-1805) | R. Murray ( ? -1799) | | M. Young (1750-1800) | W. Hales (1747-1831) |
| B. Lloyd (1782-1837) | J. Brinkley (1763-1835) | | W. Davenport ( ? -1827) | W. Magee (1766-1831) |
| D. Lardner (1793-1859) | H.H. Harte (1790-1848) | T.R. Robinson (1792-1882) | T. Luby (1800-1870) | F. Sadleir (1774-1851) |
| W.R. Hamilton (1805-1865) | J. MacCullagh (1809-1847) | | H. Lloyd (1800-1881) | |

FIGURE 2: Principal Contributors to the Dublin Mathematical Tradition

out of the events leading to this important reform are far more complex than this "lone crusade theory" suggests, and it will require much patient scholarship to unravel it in any satisfactory manner. On his promotion, Lloyd immediately introduced the works of Lacroix, Poisson and Laplace into the medal examination [20], and more significantly perhaps also into the Fellowship examinations. He was in fact continuing the reform of the mathematical curriculum for undergraduates which had already been initiated with the earlier introduction of mathematics and algebra [21]. To effect his reform, Lloyd chose the method of writing a text-book. His *Analytical Geometry, a Treatise on the Application of Algebra and Geometry for Use of Undergraduates at T.C.D.* was published in 1815 seven years after the appearance of another "reformed" classic, Brinkley's *Elements of Plane Astronomy*.

The reform movement in Cambridge had begun with the work of Robert Woodhouse, Senior Wrangler in 1795, who has been described as the apostle of the analytical movement, since his *Principles of Analytical Calculation* (Cambridge, 1803) had apparently little contemporary influence in Cambridge. The official history of Cambridge Mathematics [22] dates the beginning of the reform as 1812, with the formation of the undergraduate "Analytical Society" of Babbage, Peacock and Herschel. These Cambridge reformers produced three publications in the period 1813-1820 [23]. However, William Whewell's *Elementary Treatise in Mechanics* (1819) was, perhaps, more influential in effecting this reform as he was appointed moderator of the university mathematics examinations in 1820 [24]. In 1820 Woodhouse was appointed to the Lucasian Chair in the university, copper-fastening the reform in Newton's university.

The real distinction between Dublin and Cambridge was due to the positions of Lloyd and Brinkley as professors, while the reformers at Cambridge were outside the establishment before 1820. The Dublin reformers had in addition published reformed text-books which, significantly, were not an uncritical



acceptance of French work, but rather were works of scholarship based to a recognizable extent on a University of Dublin tradition of mathematical text-books; thus Lloyd's was a replacement in part of Hamilton's book and Brinkley's a replacement of the book of Hales and Stack. Brinkley had, in fact, obtained the Andrew's Chair in competition with Hales, although Hales had strong support in his application from Fellows of the University.

The analytical notation was introduced into the curriculum at the University of Dublin in 1815, which was five years before Cambridge. During this period Lloyd moulded the minds of the leading undergraduates to produce an analytical school in Dublin. This school arose because of Lloyd's excellent teaching and boasted among its members some outstanding young lions. Lloyd's first generation disciples included Thomas Romney Robinson (1792-1852), Denis (Dionysius) Lardner (1793-1856), H.H. Harte (1790-1848), Thomas Luby (1800-1870) and perhaps Franc Sadlier (1774-1851) who later succeeded Lloyd as Provost. Those of this group were characterized by their commitment primarily to educational works. The Dublin reformers seemed to be classic examples of consolidations in a Kuhnian paradigm [25] and they boasted with their leaders an impressive list of educational publications in the period up to 1831 (Fig. 3). Their work was rewarded by the promotion of their mentor Bartholomew Lloyd to the Provostship in 1831 and the succession of Sadlier.

The reasons for the advanced position of Dublin with respect to Cambridge are simple to guess at, but almost impossible to substantiate. Cambridge was the centre of the British Newtonian tradition, while Dublin had a long legacy of critical acceptance of this tradition beneath which there lay a Cartesian current [26]. The connections between Ireland and France were very strong in the period up until 1798, and this was seen by those of influence in England as being treasonable. The principal reason for the advanced analytical position of Dublin

1815 : B. Lloyd, *Treatise on Analytical Geometry.*
Medals in Examination awarded on a new basis.

1820 : T.R. Robinson, *System of Mechanics.*
D. Lardner, *Central Forces*

1823 : D. Lardner, *Algebraic Geometry.*

1824 : D. Lardner, *Elementary System of Mechanics.*
D. Lardner, *A Series of Lectures on Locke's Essay.*
Brinkley receives Copley Medal of Royal Scoiety

1825 : D. Lardner, *Treatise on Differential and Integral Calculus.*

1826 : B. Lloyd, *Treatise on Mechanical Philosophy.*
Brinkley promoted to See of Cloyne.

1827 : W.R. Hamilton becomes Astronomer Royal at age of 21.

1828 : T. Luby, *Physical Astronomy.*
D. Lardner, *Discourse on the Advantages of Natural Philosophy* (London).
D. Lardner, *Treatise on Plane and Spherical Trigonometry* (London).
D. Lardner, *First Six Books of Euclid* (London).

1829 : D. Lardner, *Mechanics and Pneumatics and Newton's Optics* (SDUK).

1830 : H.H. Harte, *The System of the World* (2 Vols) (London). Translation of Laplace's *Systeme du Monde.* Later translates Laplace's *Mecanique Celeste* and Poisson's *Mecanique.*
D. Lardner and H. Kater, *Treatise on Mechanics* (London).
D. Lardner, *A Treatise on Hydrostatics and Pneumatics.*

1831 : H. Lloyd, *Treatise on Light and Vision.* (SDUK).
B. Lloyd becomes Provost of T.C.D.
H. Lloyd Professor of Nat. Phil.
Moderatorship Examinations (Honours) introduced.

FIGURE 3: Chronology of Reform Text Books by University of Dublin Authors Before 1831.



was, however, almost certainly due to the profoundly middle-class nature of the University and its educational methods. The educational methods were based on text-books, lectures, grinders to support the college tutors, well defined syllabi, and competitive *viva-voce* examinations with questions requiring extremely precise responses. The examination was the sole arbiter of the degree award. This unique system led the University to be crowded with "back stairs men" at the quarterly examinations, who otherwise did not attend as they studied in their own chosen manner and time outside the University. These much despised students, however, had the supreme virtue of greatly increasing the revenue of the Fellows, whose incomes were dependent on examination fees. The nature of mathematics lent itself very well to this examination system, and no doubt Lloyd and the reformers exploited this advantage using mathematics as a wedge to open the way for their more ambitious plans whose implementation followed Lloyd's election as Provost [27].

Brinkley and Lloyd appeared to be radical reformers in 1815, but these gentlemen were not a revolutionary force. They did aspire, however, to be an intellectual leadership of the nation and were consequently enthusiastic supporters of Coleridge's idea of a "clerisy" [28]. The pattern of French secular professionalization of science was not repeated in Britain because the leaders of the reform movement such as Lloyd and Brinkley were professional only in a vocational sense. The Dublin leaders' expertise in their subject specializations gave then a real advantage in the scientific fraternity of their day which was demanding professional standards. "The gentlemen of science" as Morrell and Thackray [29] named them, were only one competing group for the leadership of the British scientific fraternity. In 1831 when Lloyd was elected Provost, the outcome of the analytical reform was still in the balance with the main battle front centred on the wave corpuscular controversy in optics. The reformers had captured the high ground in Dublin and Cambridge, but the initial battle for the Royal

Society had been lost in 1830. The provincial scientific societies, which had emerged along with the Mechanics' Institutes with the development of the industrial revolution, were largely led by Newtonians such as Edinburgh's Henry Brougham and David Brewster.

The process of professionalization in France [30] had been associated with a political struggle for ascendancy between "the savants" (theorists) and "manants" (experimentalists) and by 1815 "the savants" had effectively established an ascendancy over the "manants". In Britain the mathematicians had set out to repeat this process by rendering the new Baconian sciences mathematical. This threatened seriously to undermine their dominance, as these new sciences appeared to have at that time destroyed the dominance of the Newtonian sciences. This "analytical revolution" as it has been called was therefore a reform out of necessity for the University of Dublin, as it was threatened by the emergence of a burgeoning Baconian institute, the Royal Dublin Society and later the new Dublin Mechanics Institute from 1825. The paradox was that the practical men largely defended the old Newtonian orthodoxy which the theoreticians, who had formerly been stoutly Newtonian, now attacked.

The emergence of a new mathematical physics which developed from this ideological struggle was consequently modelled on the French 'physique'. That development has been recently investigated from an English perspective and the study identified a strong Irish involvement in this process [31]. The first group of Lloyd reformers at the University of Dublin with the two leaders, effected a thorough going change in the old Newtonian tradition of Dublin with their wide-ranging educational works. Perhaps more significantly this prepared the ground for a new second generation of mathematicians, who were differentiated from the first by their commitment to the generation of new knowledge. The change can be summed up by saying that the ideals of the Royal Irish Academy in this period gained



ascendancy over those of the Dublin Society.

## Concluding Comments

The analytical reform carried through by Lloyd was a major break with the old tradition at the University of Dublin which, as his disciple Lardner said, allowed "the study of mathematics to leap a chasm of one hundred years" [32]. Lloyd brought into the centre of Irish mathematics not only an awareness of the contemporary French works of Lagrange, Laplace, Poisson, Fourier, Monge, Legendre and Lacroix, but it also enabled the Dublin mathematicians to assimilate the earlier work of the 18th century continental mathematicians such as that of Euler and the Bernoullis.

An objective measure of Lloyd's educational reform was the 1822 undergraduate science Medal Examinations. These were based on Woodhouse's *Trigonometry*, Lardner's *Algebraic Geometry*, Lacroix's *Calcul Differentiel et Integral* and *Theorie des Lignes Courbes*, Lloyd's *Mechanical Philosophy*, Poisson's *Mecanique* and selections from Newton's *Principia* and Laplace's *Mecanique Celeste*. This curriculum was followed by the best of the undergraduates in the subsequent period. Before Lloyd's reform the undergraduates body "were employed fathoming the mysteries of Decimal Fractions" [33]. In 1822 they boasted among their number William Rowan Hamilton who was then pursuing his research into mathematical optics [34]. Lloyd's principal achievement therefore was that his work opened up for the first time in a British University the great range of continental discovery and marked the introduction to these islands of higher analysis. He led his disciples into an alliance with Cambridge based mathematicians, initially on the mutual commitment to what Babbage called the pure principle of "D-ism", that is the Leibniz notation, against that of Newton which Babbage characterized as the "dot-age", but later this alliance had other major and far-reaching implications for British mathematics.

One of Lloyd's devoted admirers, J.H. Singer, delivered the eulogy at his Memorial Service. This eulogy provided a contemporary assessment of the significance of his reforms.

> "Our University bears proof of the skill and prudence with which he could adapt the institutions of venerated antiquity to meet the demands of modern improvement, and the rapid and accelerated advance which our Institution and our country have made in all the various Departments of Science is connected essentially with the name and labours of our gifted and venerated Provost."

The Rev. Bartholomew was undoubtedly the most successful reforming "gentleman of science" of his age and it was through such reforms that Britain was transformed into a great power of the 19th century. Mathematics had as part of this process of modernization to be reformed to meet the new demands of a nation rapidly industrializing and widening continuously its spheres of activity. The only way for Britain to meet the challenge of the French in the early years of the 19th century was by emulation and Lloyd was the man at this time who organised the stealing of the French cloths and thereby prepared the way for the advance of Irish mathematics which began its first real independent flowering in the period of his short Provostship. The Dublin Mathematical School was Lloyd's creation although initially inspired by Brinkley. Brinkley was also the man who provided the model for the work of the second wave of Lloyd's reformers on whom the fame of Irish mathematics still securely rests.

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8. This work connects with the early study of Marsh in the D.P.S.
9. D.N.B. entry states that his society for the study of Syriac theology and philosophy founded in 1777 became a germ of the R.I.A.
10. Hales' scientific works were on sound (1778), planetary motion (1782), Analysis of chronology (1809-1812). He was Rector of Killeshandra from 1787.
11. In T.C.D. in manuscript form are two books in Latin: *Principorum Newtoni*, apparently prepared for publication. T.C.D. manuscript N.4. 14-17.
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16. Brinkley's work on parallax was a development of Molyneux's earlier researches.

17. McDowell and D. Webb, *Trinity College in the Age of Revolution and Reform (1794-1831)*, Hermathena, No. LXXII, 1948.

18. *op cit.*, Note 2.

19. D. Lardner, Preface, *Algebraic Geometry* (Dublin 1822). It is that in addition to Brinkley, Lloyd had the active assistance of Davenport, professor of natural philosophy and probably others such as Franc Sadleir.

20. The origin of the honours examination lies in these competitive examinations.

21. Algebra first appeared in the T.C.D. curriculum in 1808 according to R.B. McDowell and D.A. Webb, *Trinity College Dublin 1592-1952; An Academic History*, (Cambridge University Press, 1982), p. 90. Mathematics first appeared in the curriculum in 1793, p. 69, although arithmetic it was assumed was taught in schools.

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23. Cambridge reform works before 1821 were *Memoirs of the Analytical Society* (1813) Babbage and Herschel, *Translation of Lacroix Traite* (1816) Babbage, Herschel and Peacock, *A Collection of Examples of the Application of the Differential and Integral Calculus*, (1820) Peacock, Herschel and Babbage (2 vols).

24. Whewell was moderator in 1820 and was succeeded in 1821 by Peacock.

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26. The Molyneux influence in Dublin was enduring and his early Cartesianism perhaps is most clearly expressed in the Dublin tradition of vortex studies in mathematics which were central to the work of James MacCullagh in his search for the Cartesian synthesis in the 1830s and 1840s.

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*School of Science,*
*Regional Technical College,*
*Carlow.*

# TOWARDS MORE EFFECTIVE MATHEMATICS EDUCATION

*S. Close*

*"The effects of our teaching programme are frequently disastrous ---. The students have no sense of history of the subject, nor its origins. They see no relationship between mathematics and the world in which we live. They are continually confronted with definitions and theorems completely cut off from their historical and quite valid origins ... But worst of all we kill any enthusiasm our students have for the subject which we present as a logical and pedestrian development of results from an apparently arbitrary base made up of some axioms."* [4]

*"We have become convinced that a major contribution to the difficulties that students are having (in first university Physics courses) comes from their grasp, or rather lack of the fundamentals of mathematics. --- We have discovered an appalling lack of the most elementary mathematical preparation among the first year Science students."* [10]

*"You who are reading this most likely know what's wrong with secondary school mathematics in Ireland. The only question you need to have answered now is: When is something going to be done about it?"* [3]

The above comments, made by authors of articles which have appeared in recent issues of the *Irish Mathematical Society Newsletter*, suggest that there are serious problems in mathematics education at both the second and third levels of schooling. Indeed, at second-level, the Government Department of Education Syllabus Committee in Mathematics is presently reviewing the existing syllabus. This is probably as a



result of criticisms from university mathematics departments, colleges of technology, post-primary teacher associations (including the Irish Mathematics Teachers Association), and industrial bodies. Despite the lack of formal research evidence it seems clear enough that many students are failing to learn mathematics effectively in our schools and colleges at the present time. It is also apparent that many educators and employers accept this situation as inevitable and as an artifact of innate human variability and even see it as desirable for differentiation and selection purposes. For those who would seek to improve on the present system I would like to clarify some of the educational variables involved and make a few suggestions on how educational theory might be drawn upon to derive models and principles (many of which have been well researched) for addressing these variables in facilitating mathematics learning.

Writing in the 69th Yearbook of the National Society for the Study of Education in the U.S.A., Lee S. Shulman described "a model for examining those variables which must be considered in formulating any propositions about the best forms of (mathematics) instruction" [11]. A diagram of the model is presented in Fig. 1 below.

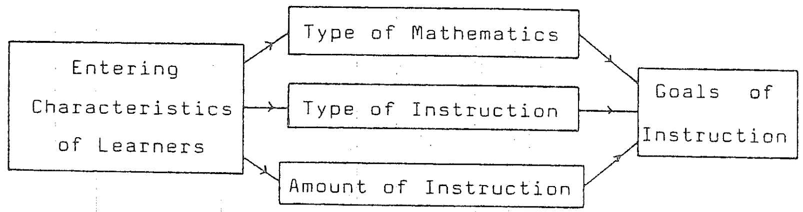


FIGURE 1: Theoretical Model of Nature of Mathematics Instruction

In accordance with this model mathematics instruction is viewed as a complex interaction of variables which can be categorised under the headings: mathematical content, learning environment, time for learning, results of learning, and, the learners. The problem of building mathematical and statistical models to describe and examine the nature of the relationships in this kind of theoretical model of the instructional process has proved to be more complex and difficult than is the case with models of physical and natural processes. The educational research literature is laden with reports of empirical studies in which carefully formulated hypotheses find no support in the data collected and analysed. On the other hand there are also many reports of studies yielding clearcut findings which can be used to develop more effective instructional principles. In a joint publication of the Mathematical Association of America and the National Council of Teachers of Mathematics, E.G. Begle [1] and his colleagues at Stanford comprehensively reviewed the (largely American) empirical literature in mathematical education in which they categorised the findings under headings somewhat similar to those of the above model. I would like to make some comments on aspects of the model based on interpretations of existing educational theory and research.

*Type of Mathematical Content*: The selection of mathematical content for courses is an aspect of mathematical education which is often fraught with controversies and difficulties. This is understandable when one considers the wide range of mathematical topics available, the wide range of abilities of learners and the rapid development of modern society. Current educational theory suggests that logical structure, psychological factors, and sociological factors should be given varying emphasis and consideration in making content decisions for various groups of students. Mathematicians would argue that as one moves further up the mathematical 'ladder' the more important the logical structure of the content becomes. The Intermediate Certificate course in geometry has come in for strong criticism due to its apparent lack of structure and cohesiveness [3].



In regard to the psychology of learning mathematics, it can be argued that, since mathematics is a highly structured subject, serious effort should be made to choose, organise and place topics so that students or groups of students can be fairly accurately located on a continuum of knowledge acquisition (or mathematical development) appropriate to the past achievements. We do not pay nearly enough attention to the notions of continuity and of maintenance of acquired knowledge in the business of syllabus/curriculum design. Many of the learning problems in second and third level courses are probably related to the lack of a number of very basic prerequisite skills and items of knowledge which could easily be incorporated into syllabi and given some attention (manipulation of algebraic expressions and of quantities expressed in scientific notation are two often-mentioned ones at first-year college level).

From a sociological viewpoint, the significance or relevance of a topic should be another consideration in making decisions about syllabus content. The relevance or significance of a topic resides in its historical origins, in its usefulness in society, in its applications in technology and in its intrinsic appeal. Finally, I would argue that, to some extent, the more attention given to logical coherence, meaningfulness, and relevance in syllabus design the more successful will students be in learning the syllabus content. It might also be added that there is a need to consider the interaction of content selection considerations and student characteristics. For example, some students seem to learn mathematics more effectively when the relevance of the topic is stressed.

*Goals of Instruction* : The specification of the desired outcomes of a course/programme in mathematics should consist of something more concise and more tangible than a listing of mathematical content topics. The inclusion of reference to the levels of cognitive complexity in relation to each content area helps to ensure a balanced coverage of the mathematics and facilitates instruction and assessment by delineating the range of tasks or class of problems under consideration. An example of a system of classification of levels of cognitive complexity which was used in the International Study of Achievment is given in Fig. 2.

Content Areas

1. Knowledge of definitions, notations.
2. Techniques and skills.
3. Translation of data.
4. Comprehension.
5. Inventiveness.

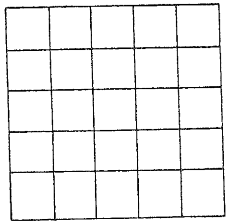

FIGURE 2: Levels of Cognitive Complexity in Mathematics

The intersection of the cognitive level and content topic defines the objective of instruction (e.g. knowledge of a technique for proving a particular theorem). A cognitive level - content area approach to goal setting would particularly suit ordinary level courses and college service courses in mathematics. Another suggestion for teachers/lecturers on such courses is that they set objective questions and performance standards that are within reach of most of the students. This suggestion is unlikely to be feasible in post-primary schools mathematics where the public examination system is designed to achieve a nice 'normal distribution' of mathematics scores with means and standard deviations so as to facilitate grading and selection and not such as to alarm the public. Perhaps the recent formation of a Curriculum and Examinations Board may herald a move towards a more beneficial approach to goal-setting and assessment.

*Type and Amount of Instruction* : Here we come to consider briefly the categories of variables which are at the hub of the mathematical education process. With the development of many alternative instructional models (e.g. discovery learning,



inductive thinking, inquiry training, advance organisers, to name a few) and a burgeoning educational technology (visual projection devices, micro-computers, T.V. and videotape, programmed texts, copying devices etc.) one might have expected that, the traditional teaching approaches of a series of lecture/chalk and talk sessions and assignments to large groups of students, would be on the wane. Not so, according to many observers. Although I cannot quote any Irish research survey it is unlikely that mathematics instruction here is any less traditional than in the U.S. where a 1977 National Science Foundation study of approximately 5,000 secondary school classrooms revealed that the predominant instructional pattern is teacher explanation followed by pupil work on class assignments [6]. It is probably fair to say that in our third-level colleges and universities, apart from some advanced mathematics courses, the predominant instructional style is still that of lecture series and occasional seminars. Such an inflexible and unresponsive learning environment promotes mediocre learning and poor study habits at all levels of mathematics education. A number of educationists [9] argue that with the advent of computer-based education the opportunity will shortly exist to provide students in many student areas, including mathematics, with a learning environment in which (1) they have much more more control over what they learn, and the rate at which learning material is presented and, over the time of instruction and assessment, (2) they obtain better and more regular feedback on performance. The instructor will be freed to do higher level work such as course development and management, Individual consultation, group discussion, etc.

*Characteristics of the Learners* : It makes good sense in providing mathematics courses to take account of individual differences. Many pupils come to mathematics courses already knowing quite a lot of the content to be learned, others come to mathematics courses without various prerequisite mathematical concepts and skills. Such differences can be easily accomodated in instructional provision but are often ignored. Other individual differences which are related to mathematics performance, such as spatial and logical reasoning [8], cognitive style [5], and internal structure of mathematical knowledge [2] are less easily accommodated but may, with further investigation, become more accessible variables.

In conclusion, I would suggest that although the provision of mathematics education is a complex process, there is scope for substantial improvement through changes in instructional variables such as those described above, a willingness to try new approaches, and a cooperative spirit among mathematics teachers at primary, secondary and third levels of education.

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## COMPUTERS IN THE TEACHING OF MATHEMATICS AT UCD


*Colin D. Walter*


In common with many other university mathematics departments, a considerable portion of our effort is involved with service teaching of first year courses. Much of this consists of rote techniques such as solving linear equations, inverting matrices, differentiating polynomial functions, and curve sketching of rational functions. These are all, with more or less ease, amenable to programming as tutorial sessions on a computer. The advantages are clear: a uniform procedure is taught to everyone, and each student can practice as much as necessary whenever he wishes. In this way the tutorial system can be extended, less dependence need be placed on tutors, and some formal tutorials can be replaced by sessions where groups of students use the computer while the tutor gives individual help. This is all at essentially no extra cost to the College.

Whereas teaching software does exist at school level, it is extremely difficult to find good university level programs. The interactive nature of such programs is often limited to a choice of several options and, after a selection is made, the problem is then completely solved without further assistance from the student. One would prefer at every possible opportunity to ask the user to think about what step should be done next and let the machine perform the requested task, adding comments on the choice.

There is the apparently opposing desire to avoid any requirement for introducing special notation, teaching programming, or knowing the operating system. In practice, this means a student can only be asked either to choose from a set of options or type in a mathematical expression in its usual form. It is catering for this latter possibility that makes good programs exceedingly large: too long for most micros and very time-consuming to write and to make work correctly.

At the present time we have two such programs*: one for solving linear equations and the other for calculating determinants. In both cases the user may enter his own problem or have the machine generate one for him. He then types in elementary row (or column) operations until the conclusion of the problem, with a remark about his progress being made at most steps. Flow charts outlining each program are given in Figs 1 and 2. As exemplified by these diagrams, all algorithms can be described using a directed graph, each node or vertex corresponding to a step, which, in a program, becomes a block of statements. The most natural size of step for these programs is to take a step as consisting of one request for input from the student together with the subsequent calculations up to the next request.



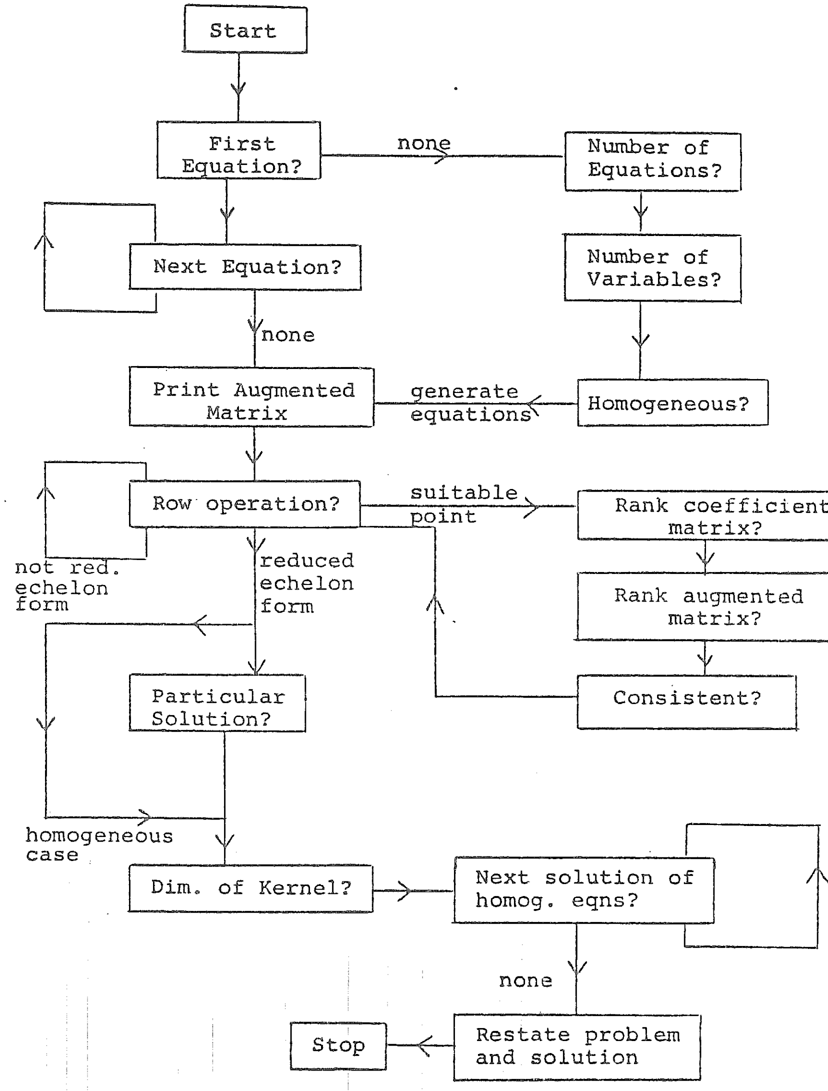


FIGURE 1: Sketch Flow Chart for "Solving Linear Equations" Program

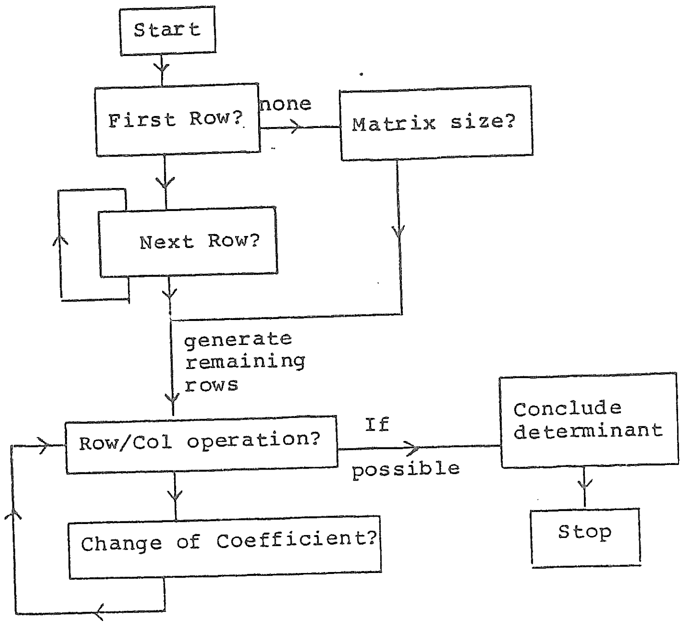


FIGURE 2: Sketch Flow Chart for "Calculating Determinants" Program

Each request from the computer can be answered either by an appropriate reply or one of a small set of letters which allow the user to move backwards or forwards through the exercise at will, ask for hints, or recap on the current state of the problem. These extra choices are as follows:

H (= Help) produces this list of letters,

F (= Forward) gives the answer to the current question and asks the next,

R (= Reverse) moves back to the previous question,

C (= Current) reproduces the present state of the problem with the current question,

I (= Information) gives a hint to answer the present question,



Q (= Quit) stops the session.

In particular, by repeatedly pressing F the student is able to obtain a complete worked example. This means that the program could be used before the material is covered in lectures. By pressing R, he can return to the previous step and make a different choice instead. The letter C is to ensure that the student is able to have sufficient information on the screen to answer a question; intervening hints or comments may have scrolled the material out of sight.

Expansion of such a service would obviously benefit students. The problems seem to be twofold. Firstly, users need a directory on a computer which allows interactive work. Restrictions on the number of terminals and on CPU time mean that careful scheduling of tutorial sessions is required. One solution is to have a login command file which logs the user off immediately if it is the wrong hour of the week and otherwise runs the program, with an automatic log-off when the program stops. This avoids any requirement to know about the system. However, examination time revision could certainly result in greater use of the programs to the detriment of the machine's performance: this is a problem of which we have no experience as yet.

The other main problem is with extending the software. The language used in the present programs is Basic, chosen because of the text editing facilities and the hope of using them on a microcomputer. A structured language such as Pascal would have been more appropriate because changes would be easier to make, existing subroutines could be used in new programs, and other authors might have a hope of adding to the existing programs.

Overall, many man-years of work are required to extend this to cover most first year material, probably several mathematicians directing some competent programmers.

It seems clear that funding for such a project would reap as benefit an improvement in our teaching so that the country would produce, for example, better engineering students and more capable scientists. At any rate the increasing abundance of available computer time is still something that we are only just beginning to appreciate and there is now a need for programmers as "laboratory technicians", not just in computer science departments but also in mathematics departments.

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* While the author is on leave of absence at the University of East Anglia, Norwich, copies of the programs are available from Dr J.B. Quigley, Mathematics Department, U.C.D., at a cost of £30 to cover the magnetic tape, packing and postage.

Minor modifications will be necessary if DEC Basic is not available.



## BOOK REVIEWS

### "SINGLE-VARIABLE CALCULUS"

By *Robert A. Adams*

Published by *Addison-Wesley*, 1983, £19.95 (sterling), 590 pp.

It must be a daunting task to set about writing a 600-page text book. It could be that the author wishes to introduce to a wider audience material not available in book form: however, in the case of a calculus text, a new author must search for new approaches, which will bring a greater unity or clarity to the subject matter. In the past twenty years, calculus texts have grown considerably in size (satisfying logistic growth rather than exponential growth, I hope), the increase in size being partly due to increase in page size for clarity in reading, but also partly due to additions of appendices to make each new edition more comprehensive. In the preface of Single-Variable Calculus, Professor Adams claims to have produced a book which "is not as massive or bulky in appearance as many other books available in recent years." Although this is certainly true, measuring $7\frac{1}{2}$"x $9\frac{1}{2}$"x $1\frac{1}{2}$" and weighing $2\frac{1}{2}$ lbs it is in no way a pocket calculus. The author is able to keep the number of pages below 600 because he regards calculus of several variables as suitable for a separate text (perhaps, he is writing a sequel himself).

Single-Variable Calculus is primarily designed for a two-semester course for science and engineering students. In Chapter 1, Functions, Limits, Continuity, the author introduces the $\varepsilon$- $\delta$ definition of a limit but relegates the proofs of the results about limits and continuous functions to an appendix. The brief treatment of inequalities I found unsatisfactory but the inverse of a 1-1 function is introduced without the confusing terminology of injections and surjections.

Some new ideas appear in Chapter 2, Differentiation: Definition, interpretation and techniques. A worked example brings out the idea of a cusp and the modulus function $|x|$ is differentiated as sgn x. Both these ideas could give rise to interesting new problems in later chapters but unfortunately the opportunity is not taken. In this chapter, the author also introduces antiderivatives, indefinite integrals, differential equations and initial value problems. This is certainly a change from the traditional treatment. It appears to succeed but for those who teach differentiation then teach integration it may be difficult to change habits of a lifetime.

Chapter 3 is devoted to teaching The Elementary Transcendental Functions. This is done at this stage so that these functions can be utilised in the next chapter. Interestingly, the author gives the pronounciation of ln as 'lawn' but gives no guide as to how to pronounce sinh, tanh etc.

Chapter 4, Various applications of Differentiation deals fully with the search for local maximum and minimum points by noting that these can occur at critical points, end points and singular points. Systematic procedures are given for solving Optimisation and Related Rates problems which end with the sound piece of advice "Make a concluding statement answering the question asked." My only quibble with this chapter is that in treating Newton's method it is good practice to show how the calculations can be laid out neatly in tabular form. This is not done.

The smallest chapter, Curves in the Plane, consisting of optional material, is followed by the longest on integration. A list of 20 integrals is given that the reader is told to memorize, undoubtedly another piece of sound advice, although certainly frowned upon by most Irish secondary school teachers. One of the handiest rules of integration is

$$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + C.$$



No mention is made of this: has it gone out of fashion?

The standard applications of integration are covered in Chapter 7, viz. Volumes, arc length, surface area, center of gravity all within the limitations of one variable calculus.

We must wait until Chapter 8 on Infinite Series to meet those dreaded words "the proof is left as an exercise." Apart from an unusual way of proving $\lim x^n = 0$ if $|x| < 1$, the treatment of infinite series is good with a section on "Estimating the sum of a series" which other calculus authors should consider adding to their next editions. The book concludes with a short chapter on Power Series representation of functions.

Interspersed throughout the text are more than 2000 problems: the vast majority provide drill in basic techniques but there are also a number of more interesting and harder asterisked problems to challenge the better student. Solutions of odd-numbered problems appear at the end of the book.

Overall, the presentation and layout of the material is excellent, up to the high standard one has come to expect from Americal text books. With a first edition, there is always the problem of errors and misprints. Although the author does suggest that these have all been removed a number of mistakes remain. These are mainly of a minor nature although the statement of Theorem 2 of Chapter 9 and a subsequent statement are false, and there is a mistake in a worked problem on L'Hôpital's rule.

A modern trend, of which I wholeheartedly approve, is to give biographical details of those whose names appear in the text (one book recently even had photographs and half-page biographies of the principal workers in the area). I certainly feel that such (trivial?) details make a text book much more friendly. Perhaps, in America the history of mathematics is sufficiently covered in other courses but here in Ireland we seem to ignore these matters. I would certainly welcome such additions to this text.

In the Preface, the author argues for the need for a separate text on calculus of several variables. Personally, I am not convinced, for economic as well as other reasons, by this argument. However, for those who feel the need for a text on Single-Variable Calculus I can certainly recommend this book.

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"TOPOLOGY AND GEOMETRY FOR PHYSICISTS"

By *Charles Nash, St. Patrick's College, Maynooth, and Siddartha Sen, Trinity College, Dublin*

Published by *Academic Press, London,* London, 1983, Stg. £31.50,

ISBN 0-12-514080-0

MATHEMATICAL PHYSICS YOUR MOTHER NEVER TAUGHT YOU

Mathematical Physics as a discipline has been defined and dominated by a single book in a way that no other field of science has been. This book is, of course, Methods of Mathematical Physics by Richard Courant and David Hilbert. Courant and Hilbert first appeared in Germany in 1924, and has been continuously available in a sequence of different forms ever since. It is still in print, the two volumes costing well over £100.



The mathematics in Courant and Hilbert, despite some modern touches, has a curiously nineteenth century flavour to it. The book is focused on differential equations and so naturally deals with continuous functions. The way it links up with the discontinuous nature of much of modern physics, especially quantum mechanics is via the eigenfunction/eigenvalue approach where each individual eigenfunction is a solution to a differential equation and so inherits its differentiability properties from it, but the eigenvalues themselves tend to be discrete.

The only significant branch of Mathematical Physics which stood apart from the Courant-Hilbert approach was group theory, which used the discreteness of the group elements to model nature. Thus, ten years ago, if one had a good grounding in both Courant/Hilbert and some group theory, all one needed was a smattering of physics and one could hold one's head up as a mathematical physicist in the fanciest of company.

This golden age has completely vanished. In the last decade there has been a flood of new ideas flowing into physics from mathematics especially in geometry and topology. For example, a unit cell in a crystal is a three-manifold without boundary. This means that it is trivial to show that the total charge in each cell must be zero. Of course, this is not a new result, but it is a very simple example of how even elementary topology can and should be used.

The book under review is an attempt to codify and make available to the ordinary physicist the key ideas of modern geometry and topology. It assumes no previous knowledge, and so starts off with two introductory chapters, one on general topology and one on differential geometry. Then come four chapters on homotopy, homology and cohomology. The last of the mathematical chapters is a long (80 page) chapter called "fibre bundles and further differential geometry". The last three chapters apply the previously developed tools to a range of physical problems. Two of these are fairly short and deal respectively with Morse theory, which is applied to phase transitions in crystals, and the theory of defects.

Pride of place, naturally, is given in the final chapter to Yang-Mills theory. Yang-Mill's theory was invented thirty years ago but was virtually ignored for twenty years. However, in the last ten years there has been an enormous investment in time and effort by the physics community to understand Yang-Mills theory and much of the recent advances in elementary particle physics have come from this work. Yang-Mills theory is what physicists call a gauge theory, which means that it has a natural geometrical structure. It is the attempt to unravel this geometrical structure that has forced physicists to learn geometry and it is the prime motivation for this book.

In their preface, the authors claim that they have attempted to strike a balance between rigor and clarity. I do feel that they have done this admirably. At no point during my reading of the book did I feel that the authors were sliding past me, or trying to persuade me of something, rather than proving it. At the same time, they had no hesitation in proving only part of a theorem rather than slogging through the whole thing. This means that at every stage in the book, the reader should have a very good idea of the difficulties encountered and of the techniques used to overcome these, without having to absorb a mass of material. This, I feel, is a great strength because it generates a feeling of confidence in the reader, which enables him or her actually to apply the ideas, without feeling permanently ill-at-ease.

On the other hand, this is not a book for bedtime reading. In the 300 pages is an enormous range of new ideas; not only new ideas but a new vision, a new way of seeing things. This sort of shift is not something that comes easily. When I first got the book, I was interested in learning something about Morse theory and so immediately turned to Chapter 8. I gave up very quickly. The whole structure of the book is pyramidal. Each chapter depends on the ones before, with none of the little refreshers which help reinforce the new ideas. It might be worthwhile for the novice, before plunging in, to read something easier, for example Geometrical Methods of Mathematical Physics by Bernard Schutz (Cambridge University Press, 1980).



The authors were not well-served by their lay-out man. In many places throughout the book relevant illustrations are misplaced. The printing, on the other hand, is very good, with very few mistakes in spelling or grammar or, even more importantly, in the equations.

In conclusion, I would love to teach a course with this book as a text. Both I, and the students, would learn a great deal. I can recommend it unhesitatingly. I am sure that my copy will be read and re-read in years to come.

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## PROBLEMS

Jim Stack from Waterford R.T.C. sent a solution to the parking problem (Issue No. 8) which arrived just too late for Issue No. 9. His solution was similar to the one given last time and he mentions that the idea is to be found in an article by D.E. Knuth ("Computer Science and Its Relation to Mathematics", *Amer. Math. Monthly*, April 1974) which discusses the retrieval of information from memory locations, using the analogy of musical chairs instead of car-parking.

Now for solutions to the most recent problems.

1. *The Plank Problem.* Does there exist a positive integer n such that a closed disc of diameter 1 can be covered by fewer than n planks of width $\frac{1}{n}$ ?

To see that the answer is 'no', recall that when two parallel planes meet a sphere of diameter 1 the area between the planes on the surface of the sphere is $\pi d$, d being the distance between the planes.

If the original disc is taken to lie on the equator of a sphere then the vertical projection of any plank of width $\frac{1}{n}$ meets the sphere's surface in a region with area at most $\frac{\pi}{n}$. If a collection of planks covers the disc then their vertical projections cover the surface of the sphere and so at least n planks are required.

*Remark.* The "generalised plank problem" deals with convex sets with diameter 1 (i.e. the smallest door they can be pushed through has width 1), and the planks need not have equal widths. It is to be shown that the sum of the plank widths is at least 1 and this was done by Bang, but I'm afraid I've mislaid the reference.



2. *The Planet Problem.* A finite number of equal spherical planets are in outer space. A region on the surface of one of the planets is called hidden if it is invisible from any of the other planets. Find the total area of the hidden regions.

In fact their total area is equal to the area of the surface of a single planet. This is much easier to visualise than to write down - but here goes.

Suppose that the planets have radius 1 and let $\underline{p}, \ldots, \underline{p}_n$ denote their centres (I shall identify position vectors and points throughout). Let $S_0 = \{\underline{u} : |\underline{u}| = 1\}$ so that

$$S_i = \{\underline{p}_i + \underline{u} : |\underline{u}| = 1\}, \quad i = 1, \ldots, n,$$

denote the surfaces of the planets. Also let $S = \bigcup_{i=1}^{n} S_i$, and for each $\underline{u} \in S_0$ put

$$[\underline{u}] = \{\underline{p}_i + \underline{u} : i = 1, \ldots, n\}.$$

The proof consists of noticing the following facts.

(i) The set $E = \{\underline{u} : \underline{p}_i + \underline{u}$ is visible from $\underline{p}_j + \underline{u}$, some $i \neq j\}$ lies in a finite union of circles on $S_0$, and so has area zero.

(ii) A point $p \in S$ is hidden $\Leftrightarrow$ there is a plane $\pi$ through $\underline{p}$ such that $S \setminus \{\underline{p}\}$ lies entirely to one side of $\pi$.

(iii) For any $\underline{u} \in S_0$ there is a plane $\pi$ orthogonal to $\underline{u}$ and a non-empty set $F \subseteq [\underline{u}]$ such that $S \setminus F$ lies entirely to one side of $\pi$. If $|F| > 1$ then $\underline{u} \in E$ and if $|F| = 1$ then $F = \{\underline{p}\}$ where $\underline{p}$ is hidden.

(iv) Each set $[\underline{u}]$ contains at most one hidden point. Thus, for each $\underline{u} \in S_0 \setminus E$ the set $[\underline{u}]$ consists of exactly one hidden point and the proof is complete.

*Remark.* It looks as if the result remains true when the spheres have different radii, if we replace "areas" by "solid angles".

3. *Group Theory Problem* from September 1983 Issue (page 74). Let $n$ be a natural number. A group $G$ is said to be $n$-Abelian if $(ab)^n = a^n b^n$ for all $a,b \in G$. Find all the values of $n$ for which

(i) $G$ is $n$-Abelian implies that $G$ is $(n+1)$-Abelian.

(ii) $G$ is $n$-Abelian implies that $G$ is $(n-1)$-Abelian.

*Solution* (i) $G$ is $n$-Abelian implies that $G$ is $(n+1)$-Abelian if and only if $n = 2$ or $3$.

*Proof* Clearly $G$ is 2-Abelian if and only if $G$ is Abelian. All groups are 1-Abelian, so $G$ is 1-Abelian does not imply that $G$ is 2-Abelian. Also trivially if $G$ is 2-Abelian, $G$ is 3-Abelian.

Now assume that $G$ is 3-Abelian. Then $(ab)^3 = a^3b^3$, so by cancellation $(ba)^2 = a^2b^2$ for all $a,b \in G$. Then

$$(ab) = [(ab)^2]^2 = (b^2a^2)^2 = a^4b^4 \text{ , so } G \text{ is 4-Abelian.}$$

Next assume that $n > 3$. We claim that for each such $n$ there is a group which is $n$-Abelian but not $(n+1)$-Abelian. Let $G$ be a non-Abelian group of exponent $n-1$. Then $(ab)^{n-1} = a^{n-1}b^{n-1}$ and $(ab)^n = ab = a^nb^n$, for all $a,b$ $G$. Thus $G$ is $(n-1)$-Abelian and $n$-Abelian. If $G$ is $(n+1)$-Abelian, then $G$ is $k$-Abelian for three consecutive values of $k$, so by a well-known result, $G$ is Abelian, a contradiction.

(ii) The above example shows that $G$ $n$-Abelian implies $G$ is $(n-1)$-Abelian only in the trivial cases $n = 2$ and $n = 1$.

Just one new problem this time from me:

*Problem.* For $1 \leq p \leq 2$ show that

$$(1+x^2)^p \leq 1 + (2^p - 2)x^p + x^{2p}, \quad x \geq 0.$$

What happens for other values of $p$ ?



This inequality (which I learnt from Jim Clunie) is related to a problem in $\ell^p$ spaces posed by Finbarr Holland. It is a very special case of a conjectured inequality in $n$ variables, which I'll say more about next time.

Finally, a short test on Linear Analysis and Ring Theory from Robin Harte.

Problem 1 Suppose $T:X \to Z$ and $S:Y \to Z$ are bounded linear mappings between normed linear spaces, and write row$(T,S)$ for the mapping

$$(x,y) \to Tx+Sy : X \times Y \to Z.$$

(a) If $T$ and $S$ are bounded below and row$(T,S)$ is one-one, does it follow that row$(T,S)$ is bounded below?

(b) If $\alpha > 0$ and $\beta > 0$ are such that, for each $x,y \subset X,Y$,

$$||x|| \leq \alpha||Tx|| \text{ and } ||y|| \leq \beta||Sy||,$$

and if row$(T,S)$ is one-one, does it follow that

$$\max(\,||x||\,,\,||y||\,) \leq ||\alpha Tx + \beta Sy|| \text{ ?}$$

Problem 2 Suppose $T:X \to Y$ and $S:Y \to Z$ are bounded linear mappings between normed linear spaces:

(a) if $ST$ is one-one, $T$ is bounded below and $S$ is relatively open, does it follow that $ST$ is bounded below?

(b) if $\alpha > 0$ and $\beta > 0$ are such that, for each $x,z \in X,Z$,

$$||x|| \leq \alpha||Tx|| \text{ and } z \in S(Y) \Longrightarrow z \in \{Sy : ||y|| \leq \beta||z||,$$

and if $ST$ is one-one, does it follow that $ST$ is bounded below?

Problem 3 Suppose $A$ is a ring, with identity 1, and $a_1$, $a_2$, $b_1$, $b_2 \in A$ satisfy

$$b_1a_1 = 1 = a_2b_2 :$$

does it follow that $a_2a_1ca_2a_1 = a_2a_1$, with possibly $c = b_1b_2$ ?

Problem 4  Suppose A and B are rings with identity 1, and suppose that $T:A \to B$ is additive and satisfies

$$T(1) = 1 \text{ and } T(A^{-1}) \subseteq B^{-1},$$

where $A^{-1}$ and $B^{-1}$ are the groups of invertible elements in A and B. Does it follow that T is multiplicative? Does it at least follow that T has the Jordan property

$$T(a_2a_1+a_1a_2) = T(a_2)T(a_1) + T(a_1)T(a_2) \ ?$$

Problem 5  Suppose A is a ring with identity, and write $A_{n \times n}$ for the matrices over A. If $a \in A_{n \times n}$ has mutually commuting entries $a_{ij} \in A$ and a left inverse $b \in A_{n \times n}$, must its determinant $|a|$ have a left inverse in A?

*Phil Rippon,*
*Faculty of Mathematics,*
*The Open University,*
*Milton Keynes.*



## A SHORT DICTIONARY OF MATHEMATICAL TERMS

*Des MacHale*

As a young graduate student I was frequently perplexed by certain words and phrases which cropped up again and again in the research papers which I attempted to read. Conventional Mathematics dictionaries gave me no help whatsoever, but experience has since taught me the true meaning of many of these expressions. For the benefit of those who find themselves in the same position, I offer a selection, in the hope that it will stimulate others to contribute to this sadly neglected area of mathematical education.

1. *The proof is left as an exercise*: I've lost the envelope on which I jotted this down, but it seemed reasonable at the time.

2. *While the results of Holland are relatively deep*: Holland once mentioned a paper of mine in his references.

3. *Formal Process*: I can't understand this for the life of me, but it seems to work.

4. *By far the most significant results in this field are due to Hurley*: Hurley is likely to referee this paper.

5. *I wish to thank the referee for a number of useful suggestions*: The old meanie cut me down from twenty pages to a miserable four.

6. *While only partial results have been obtained*: I've made no progress at all with this problem but I figured I could get at least one publication from it.

7. *It is well known that*: I'm not quite sure how to prove this and I'm hanged if I'm going to the trouble of finding out who first discovered it.

8. *Harte (oral communication) has shown that:* I cornered him and bored him to tears during a coffee break at a recent conference.

9. *A straightforward calculation gives:* A very difficult calculation, which took me the best part of a week, gives.

10. *Without loss of generality:* I can't handle the general case at all.

11. *Which completes the proof:* Which completes the proof, I hope.

12. *Evidently, clearly, obviously:* Maybe.

13. *Using a deep result of Vernon:* Vernon's work is completely beyond me, but I know a useful theorem when I see one.

14. *An interesting comparison might be made between the present results and those of Barry:* There is no connection at all, but his name looks great in my references.

15. *On pseudo-compact semiheaps with involution I:* I hope to get at least four papers out of this useless and obscure topic.

16. *This problem is of great theoretical significance:* I'm the only one who is interested in it or knows anything about it.

17. *I wish to thank Miss Sheehan for her patience and excellent typing:* She has threatened never to type another word unless I put this in.

18. *I wish to thank Dr. Seda for some valuable suggestions:* All the ideas and work are due to Dr. Seda, my supervisor.

19. *It is natural to ask the question:* One of my research students just has.

20. *We are sorry that due to lack of space we cannot publish your article:* Some people have a neck sending such rubbish



to a distinguished journal like ours.

21. *Some of his results are in conflict with ours:* The guy's crazy. SL(2,p) is an obvious counterexample.

22. *Dr. Fitzpatrick has kindly pointed out an error in Lemma 3:* Why doesn't that ***** mind his own business.

23. *I wish to thank my supervisor for his valuable assistance in the preparation of this paper:* I saw him once in the distance at a conference.

*Department of Mathematics,*
*University College,*
*Cork.*

## CONFERENCE ANNOUNCEMENTS

### FOURTH CONFERENCE ON APPLIED STATISTICS IN IRELAND

This conference will be held at Kilkea Castle Hotel, Castledermot, County Kildare on March 29-30, 1984.

### *CONFERENCE PROGRAMME*

*THURSDAY 29 MARCH*

| | | |
|---|---|---|
| 12:00- 2:00 | Registration and Lunch | |
| 2:00- 3:45 | *SESSION ONE* | |
| | 1. Design of Model Discriminatory Pharmacokinetic Experiments | A. Dunne (UCD)<br>L. Lacey (May & Baker Ltd) |
| | 2. Practical Problems in the Statistics of Image Processing | G. Horgan (TCD) |
| | 3. Maximum Likelihood Discriminant Analysis on the Plane Using a Markovian Model of Spatial Context | J. Haslett (TCD) |
| | 4. Computer-Mapping of the Census of Agricultural Statistics | A. Horner (UCD)<br>J. Walsh (Carysfort) |
| 3:45- 4:15 | Refreshments | |
| 4:15- 5:00 | An Introduction to the Central Statistics Office and Its Activities | T.P. Linehan (Director, CSO) |
| 5:10- 6:25 | *SESSION TWO* | |
| | 5. Marks and Standards | A. Moran (UCC) |
| | 6. Teacher Assessment of Pupil Activity: A Regression Model | O. Egan (ERC) |
| | 7. Some Uses of the Micro-Computer in the Teaching of Statistics | E. McEntee (UP) |
| 6:25- 7:00 | Demonstration of Computer Equipment | |
| 8:30 | Dinner | |



*FRIDAY 30 MARCH*

| | | |
|---|---|---|
| 9:00-10:45 | *SESSION THREE* | |
| | 8. Maximum Likelihood Estimation in Electrophoretic Studies | S. Gardiner (QUB) |
| | 9. Application of a Special Case of ANOVA in the Testing of a Statistical Information System | D. McSherry (QUB) |
| | 10. The Application of Psychological Data to Multivariate Analysis | P. Whalley (OU) |
| | 11. The Influence of Measurement Errors in Multivariate Analysis | E. Gillespie (UP) |
| 10:40-11:10 | Refreshments | |
| 11:10-11:55 | The Role of Statistics at Guinness | C. Smith<br>A. Reynolds } Guinness |
| 12:00-12:50 | *SESSION FOUR* | |
| | 12. A New Continuous Multivariate Exponential Distribution with Applications to Reliability, Computer Breakdowns and Data Analysis | A. Raftery (TCD) |
| | 13. A Sequential Design in Attribute Testing | A. Yazdi<br>M. Khan (KSU) |
| 1:00- 2:30 | Lunch | |
| 2:30- 3:45 | *SESSION FIVE* | |
| | 14. Does Random Sampling Affect Irish Elections? | A. Unwin (TCD) |
| | 15. Non-Parametric Regression | D. Barry (UCC) |
| | 16. Bootstrapping a Regression Equation | G. Kelly (UCC) |
| 3:45- 4:15 | Closing Remarks and Refreshments | |

The Conference Fee of £60 covers meals and lodging plus the Registration Fee. In addition to a display of computer equipment, it is hoped to have an exhibition of some statistical books.

For further details, consult:

*Dr P.J. Boland,*
*Department of Mathematics,*
*UCD, Belfield,*
*Dublin 4.*

MATRIX THEORY AND ITS APPLICATIONS

A conference with the above title will take place at University College Dublin on March 22-24, 1984.

The following persons are expected to speak:

Professor G.N. De Oliveira (Coimbra, Portugal)
*Matrices Over Finite Fields*

Professor H. Wimmer (Wurzburg)
*The Algebraic Riccati Equation*

Dr L. Fletcher (Salford)
*Recent Results on Pole Assignments in Descriptor Systems*

Dr R. Gow (U.C.D.)
*Factorization of Matrices*

Professor F. Holland (U.C.C.)
*(Title to be announced)*

Professor T.J. Laffey (U.C.D)
*Integer Matrices*

Dr D. Lewis (U.C.D.)
*Hermitian Forms and von Neumann Regular Matrices*

Dr R. Timoney
*Reinhardt Decompositions of Operator Matrix Spaces*

Professor K. Seitz (Budapest)
*On Matrix Quasigroups*



Further contributions are sought and persons wishing to speak are asked to inform the organisers as soon as possible.

A report of the conference, including short accounts of the papers presented, will appear in the journal *Linear Algebra and Its Applications*.

All those who expect to attend the conference are requested to inform the organisers without delay. The conference fee will be £8.00, which will include the cost of refreshments.

Organisers: *F.J. Gaines and T.J. Laffey,*
*Mathematics Department,*
*University College Dublin,*
*Belfield,*
*Dublin 4.*

GROUPS IN GALWAY

This conference will take place on 11-12 May at University College, Galway. The speakers will include Dr P. Fitzpatrick (University College, Cork), Dr T. Hurley (University College, Galway) and Dr D. Lewis (University College, Dublin).

Those who wish to present short communications are asked to get in contact, as soon as possible, with Dr T.C. Hurley, from whom further details are available.

BRITISH MATHEMATICAL COLLOQUIUM

This conference will be held at the University of Bristol on 9-13 April, 1984.

PROGRAMME

Tuesday, 10th April

| | |
|---|---|
| 9.30 - 11.00 | M.J. Dunwoody, W. Jackson, S.C. Power, N.H. Bingham |
| 11.50 - 12.40 | C.A. Rogers (UCL) |
| 5.00 - 6.00 | J.P. Serre (Paris) - CURVES OVER FINITE FIELDS |

Wednesday, 11th April

| | |
|---|---|
| 9.30 - 11.00 | R.B.J.T. Allenby, E.A. Thompson, P.G. Dixon, C.M. Maclachlan |
| 11.50 - 12.20 | E. Rees (Edinbirgh) |
| 5.00 - 6.00 | M.O. Rabin (Harvard and Jerusalem) - THE USES OF RANDOMIZATION |

Thursday, 12th April

| | |
|---|---|
| 9.30 - 11.00 | D. Fowler, D. Kirby, R.J. Steiner, F.R. Drake |
| 11.50 - 12.40 | J.H. Conway (Cambridge) |
| 5.00 - 6.00 | H. Furstenburg (Jerusalem) - ERGODIC THEORY AND DIOPHANTINE PROBLEMS |

The early morning programme will be divided into two pairs of concurrent lectures, each of forty minutes. In the afternoon splinter groups on various branches of mathematics will be held and it is hoped that some will include a discussion and problem session. Members are invited to contribute short pap-



ers and problems.

On Wednesday evening there will be an education forum on the topic 'The Use of Computers in University Mathematics Teaching' led by D.O. Tall, and on the remaining two evenings some mathematical films will be shown.

The membership fee for the Colloquium is £12 (this includes the cost of refreshments). This will be increased to £18 for applications received after 31st January 1984.

The charges for meals and accommodation (in Hiatt Baker Hall, Stoke Bishop) are as follows:

Bed and Breakfast £8.25, Lunch £3.50, Evening meal £4.50.

(The charge for bed and breakfast includes the cost of a (self-service) late evening drink and the coach trips between the Hall and the University.) The charge for the full Colloquium will be £61.50, shorter stays will be charged on a pro-rata basis. Please note - Any alteration on bookings must be received before 25th March 1984.

*School of Mathematics,*
*University Walk,*
*Bristol, BS8 1TW.*

*H.E. Rose,*
*Colloquium Secretary,*
*18 August, 1983.*

## ST ANDREWS COLLOQUIUM 1984

Under the auspices of the Edinburgh Mathematical Society, a Colloquium will be organised by and held at the University of St Andrews from 25 July to 4 August, 1984.

The morning sessions will consist of the following courses (each of about seven lectures):

Professor J.M. Ball : *The Calculus of Variations and Non-Linear Elasticity*

Professor F.E.P. Hirzebruch : *Algebraic Surfaces*

Professor D.S. Passman : *Infinite Group Rings and Crossed Products*

Afternoon sessions will include the following series of seminars (which will run concurrently):

ALGEBRA directed by Dr P.F. Smith

ANALYSIS directed by Dr J.R.L. Webb

In addition, the first series of Copson Memorial Lectures will be given by

Professor W.K. Hayman, F.R.S.

*100 Years of Value Distribution Theory*

The registration fee is TWO POUNDS per person, *which will be increased to* FOUR POUNDS *per person for applications received after 1 May 1984.* Registration fees are payable at the time of application for membership and are non-returnable in the event of subsequent cancellations.

The following fees are payable on arrival:

| | |
|---|---|
| Membership fee : | *18 pounds per person,* |
| Accommodation fee : | *135 pounds per person.* |



The accommodation charge includes the cost of three main meals, morning coffee, afternoon tea, and all gratuities. The same fees will apply to accompanying relatives and friends of mathematicians. A limited amount of accommodation is available for the children of members of the colloquium. The Colloquium Secretary will advise on special rates for young children.

The membership fee for non-resident members of the colloquium will be 23 POUNDS per person which will include the cost of morning coffee and afternoon tea.

The colloquium timetable will be arranged so that members will be able to enjoy not only various social functions but also the amenities which make St Andrews one of our most attractive holiday resorts. In the vicinity of Scotland's oldest University are to be found five golf courses (including the celebrated Old Course), a two-mile sand beach which provides excellent sea-bathing, tennis courts (in the immediate vicinity of the Mathematical Institute) and many buildings of historical interest, while the adjoining countryside offers many pleasant walks.

*Early registration would be much appreciated.* Application forms may be obtained by sending a stamped self-addressed envelope to the Colloquium Secretary:

*Dr Dorothy M.E. Foster,*
*Mathematical Institute,*
*University of St Andrews,*
*North Haugh,*
*St Andrews KY16 9SS,*
*Fife,*
*Scotland.*

For telephone enquiries, ring 0334 76161, Ext. 8198.

## THE IRISH MATHEMATICAL SOCIETY

### Instructions to Authors

The Irish Mathematical Society seeks articles of mathematical interest for inclusion in the *Newsletter*. All parts of mathematics are welcome, pure and applied, old and new. Articles of an expository nature are preferred.

In order to facilitate the editorial staff in the compilation of the *Newsletter*, authors are requested to comply with the following instructions when preparing their manuscripts.

1. Manuscripts should be typed on A4 paper and double-spaced.

2. Pages of the manuscript should be numbered.

3. Commencement of paragraphs should be clearly indicated, preferably by indenting the first line.

4. Words or phrases to be printed in capitals should be doubly underlined, e.g.

   Print these words in capitals → Print THESE WORDS in capitals

5. Words or phrases to be italicized should be singly underlined, e.g.

   Print these words in italics → Print *these words* in italics

6. Words or phrases to be scripted should be indicated by a wavy underline, e.g.

   Print these words in script → Print *these words* in script

7. Diagrams should be prepared on separate sheets of paper (A4 size) in black ink. Two copies of all diagrams should be submitted: the original without lettering, and a copy with lettering.

8. Authors should send two copies of their manuscript and keep a third copy as protection against possible loss.

If the above instructions are not adhered to, correct reproduction of a manuscript cannot be guaranteed.

Correspondence relating to the *Newsletter* should be addressed to:


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