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Pairs of Quadratic Forms over the Real Numbers

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Abstract. This survey paper examines several topics concerning pairs of quadratic forms with real coefficients. We state a theorem that characterizes pairs of real quadratic forms having a nontrivial common zero and give a proof using a method based on point-set topology. This proof relies on determining when various subsets associated with one quadratic form are path-connected. Additionally, we describe how the signature and rank of a quadratic form change over a 2-dimensional family of quadratic forms. Finally, we delve into nonsingular pairs of quadratic forms, simultaneous diagonalization, and provide a proof of the spectral theorem. This paper presents a self-contained exposition of these results.

1. Introduction

Determining whether a quadratic form f with real coefficients has a nontrivial real zero is straightforward. One can start by diagonalizing f using linear algebra techniques or repeatedly applying the completing the square method to express f in the form $d_1X_1^2 + \cdots + d_nX_n^2$. Then, f has a nontrivial real zero if and only if the d_i's are not all positive and not all negative.

Suppose that $f, g \in \mathbb{R}[X_1, \ldots, X_n]$ are quadratic forms in n variables. How does one determine if f, g have a nontrivial common real zero? The answer to this problem is much harder, but it is well known to experts and has been discussed in many places. See [8] for a large bibliography on this subject. One of the main goals of this paper is to answer this question with an exposition that is as self-contained as possible.

In Proposition 4.4, we determine when a pair of quadratic forms with real coefficients has a nontrivial common zero. We follow a method based on point-set topology that Swinnerton-Dyer used in [7, Lemma 1 (i)]. The ideas even go back at least as far as [2]. However, there seems to be a gap in Swinnerton-Dyer's proof. Our exposition will fill in the details of this gap. See Remark 4.7 for specifics.

In Section 2, we present essential material on quadratic forms that we need for this paper. Since it is no extra trouble, we give definitions, statements, and proofs of results in this section that are valid over any field K of characteristic different from 2. We introduce the objects using a basis-free approach and then routinely use convenient bases for efficient calculations.

In Section 3, we investigate some topological properties of the zero sets of one quadratic form with real coefficients. Some of these properties are used in Section 4 to help us solve our problem for pairs of quadratic forms with real coefficients.

In Sections 5 through 8, we study several other topics that are relevant to pairs of quadratic forms over the real numbers. In Section 5, we study how the signature of a quadratic form changes over a 2-dimensional family of quadratic forms. In Section 6,

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we study the ranks of quadratic forms in a 2-dimensional family of quadratic forms, and we relate the rank to the multiplicity of a zero in a naturally occurring polynomial. For quadratic forms $f, g \in \mathbb{R}[X_1, \dots, X_n]$, we also study in Proposition 6.4 the problem of finding a real linear combination $\lambda f + \mu g$ that splits off as many hyperbolic planes as possible. In Section 7, we study nonsingular pairs of quadratic forms and conditions when two quadratic forms can be simultaneously diagonalized. In Section 8, we apply results from Section 7 to pairs of quadratic forms over $\mathbb{R}$. In particular, we use Proposition 8.1 to strengthen a result of Heath-Brown in [3, Lemma 12.1]. See Remark 8.2 for specifics. We end by using our results to give a proof of the Spectral Theorem.

Here are some of the notations and basic notions used throughout this paper. Let K be a field and let $K^\times = K\backslash\{0\}$. We let $\operatorname{char} K$ denote the characteristic of K. Let K^{alg} denote an algebraic closure of K. Unless otherwise noted, we work only with fields K with $\operatorname{char} K \neq 2$. We let $K[X_1, \dots, X_n]$ denote the polynomial ring in n variables.

For $f, g \in K[X_1, \dots, X_n]$, we write $f \mid g$ if f divides g in $K[X_1, \dots, X_n]$. Recall that $f \mid g$ in $K[X_1, \dots, X_n]$ if and only if $f \mid g$ in $K^{alg}[X_1, \dots, X_n]$. A polynomial $f \in K[X_1, \dots, X_n]$ is a homogeneous form of degree m, $m \geq 0$, if each monomial in f has degree m. If f is a homogeneous form, we say that $(a_1, \dots, a_n)$ is a nontrivial zero of f if $f(a_1, \dots, a_n) = 0$ and some $a_i \neq 0$. A quadratic form is a homogeneous form of degree 2. We let $e_1, \dots, e_n$ denote the standard basis of K^n.

We let $\mathbb{R}$ denote the field of real numbers and $\mathbb{C}$ the field of complex numbers. Recall that $\mathbb{C}$ is an algebraically closed field and that $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$, and so $\mathbb{C} = \mathbb{R}^{alg}$.

2. Basic results about quadratic forms

Definition 2.1 (Quadratic Map)**.** Let V be a finite-dimensional vector space over a field K. A quadratic map $f : V \to K$ is a function satisfying the following two conditions.

(1) $f(av) = a^2 f(v)$ for all $v \in V$ and $a \in K$.
(2) The function $B_f : V \times V \to K$ defined by $B_f(v, w) = f(v + w) - f(v) - f(w)$ is a symmetric bilinear form.

We recover the usual notion of a quadratic form by introducing a basis $\{v_1, \dots, v_n\}$ of V. The definition of a quadratic map implies that $f(X_1v_1 + X_2v_2) = f(v_1)X_1^2 + B_f(v_1, v_2)X_1X_2 + f(v_2)X_2^2$. A straightforward induction implies that for variables $X_1, \dots, X_n$, we have

$$f(X_1v_1 + \cdots + X_nv_n) = \sum_{i=1}^{n} f(v_i)X_i^2 + \sum_{1\leq i<j\leq n} B_f(v_i, v_j)X_iX_j.$$

Let

$$f = \sum_{i=1}^{n} a_{ii}X_i^2 + \sum_{1\leq i<j\leq n} a_{ij}X_iX_j \in K[X_1, \dots, X_n],$$

where $a_{ii} = f(v_i)$, $1 \leq i \leq n$, and $a_{ij} = B_f(v_i, v_j)$ for $1 \leq i < j \leq n$. We call f the quadratic form associated to the quadratic map $f : V \to K$ and the basis $\{v_1, \dots, v_n\}$. We see that f is a homogeneous form having degree 2, as expected.

Let $f = \sum_{i=1}^{n} a_{ii}X_i^2 + \sum_{1\leq i<j\leq n} a_{ij}X_iX_j \in K[X_1, \dots, X_n]$ be a quadratic form. Associated to f is an $n \times n$ symmetric matrix $M = (m_{ij})$ where

$$m_{ij} = \begin{cases} a_{ii} & \text{if } i = j \\ \frac{1}{2}a_{ij} & \text{if } i < j \\ \frac{1}{2}a_{ji} & \text{if } i > j. \end{cases}$$

We have $f(X_1, \ldots, X_n) = X^t M X$, where $X = (X_1, \ldots, X_n)^t$. It is convenient to regard f as a function $f : K^n \to K$. For a subspace $W \subseteq K^n$, we let $f|_W$ denote the restriction of f to W.

The associated symmetric bilinear form to f is given by $B_f : K^n \times K^n \to K$ defined $B_f(v, w) = v^t M w$ where $v, w \in K^n$ are column vectors. Thus $f(X) = B_f(X, X)$. For a subspace $W \subseteq K^n$, we define the orthogonal complement

$$W^\perp = \{v \in K^n \mid B_f(v, w) = 0 \text{ for all } w \in W\}.$$

It is easily verified that $W^\perp$ is a subspace of K^n. We write $W^{\perp_f}$ if we need to specify the orthogonal complement of W for a particular quadratic form f.

Suppose that $f : K^n \to K$ is a quadratic map and that W is a subspace of K^n such that $K^n = W \oplus W^\perp$. Let $\{v_1, \ldots, v_j\}$ and $\{v_{j+1}, \ldots, v_n\}$ be bases for W and $W^\perp$, respectively. Then the quadratic form associated to f and the basis $\{v_1, \ldots, v_n\}$ is $f_1(X_1, \ldots, X_j) + f_2(X_{j+1}, \ldots, X_n)$, where f_1 and f_2 are the quadratic forms associated to $f|_W$ and $f|_{W^\perp}$, respectively, and the bases $\{v_1, \ldots, v_j\}$ and $\{v_{j+1}, \ldots, v_n\}$.

Suppose $V = W \oplus Y$ and let $f : V \to K$ be a quadratic map. Let $g = f|_W$ and $h = f|_Y$. If $B_f(w, y) = 0$ for all $w \in W$, $y \in Y$, then we write $f = g \perp h$.

For quadratic forms $f, g \in K[X_1, \ldots, X_n]$, the K-pencil of f, g, denoted by $\mathcal{P}_K(f, g)$, consists of all linear combinations $af + bg$ where $a, b \in K$, not both zero.

Note that f, g have a nontrivial common zero over K if and only if $rf + sg$ and $tf + ug$ have a nontrivial common zero over K where $r, s, t, u \in K$ and the matrix $\begin{pmatrix} r & s \\ t & u \end{pmatrix}$ is invertible. Because of this, it is often useful to replace f, g with two other convenient quadratic forms in $\mathcal{P}_K(f, g)$.

Two quadratic forms $f, g \in K[X_1, \ldots, X_n]$ are equivalent, written $f \cong g$, if there exists an invertible $n \times n$ matrix A with entries in K such that $f(X) = g(AX)$, where $X = (X_1, \ldots, X_n)$. In this situation, we say that f is obtained from g by an invertible linear change of variables. If $g(X_1, \ldots, X_n) = X^t N X$, then this is equivalent to the condition $M = A^t N A$. A quadratic form f is equivalent to a diagonal form $d_1 X_1^2 + \cdots + d_n X_n^2$ under an invertible linear change of variables over K because for any symmetric matrix M there is an invertible matrix A such that $A^t M A$ is a diagonal matrix. Such a diagonal form is denoted by $\langle d_1, \ldots, d_n \rangle$.

Definition 2.2 (Rank of a Quadratic Form over K)**.** The rank of a quadratic form f, denoted by $\operatorname{rank}(f)$, is the rank of the matrix M.

If f is equivalent to $\langle d_1, \ldots, d_n \rangle$, then $\operatorname{rank}(f)$ is the number of nonzero d_i.

Definition 2.3 (Radical of a Quadratic Form)**.** Let $f \in K[X_1, \ldots, X_n]$ be a quadratic form with associated symmetric bilinear form B_f. The radical of f over K is the subspace

$$\operatorname{rad}(f) = \{v \in K^n : B_f(v, K^n) = 0\}.$$

We say that a quadratic form f is nonsingular if $\operatorname{rad}(f) = 0$, and is singular if $\operatorname{rad}(f) \neq 0$.

We can write $K^n = V \oplus \operatorname{rad}(f)$ for some subspace $V \subseteq K^n$, and it is straightforward to check that $f|_V$ is nonsingular. We let $\operatorname{Null}(M)$ denote the null space of a matrix M.

Lemma 2.4. *Let $f \in K[X_1, \ldots, X_n]$ be a quadratic form with associated $n \times n$ symmetric matrix M.*

(1) $\operatorname{rad}(f) = \operatorname{Null}(M)$.
(2) $\operatorname{rank}(f) + \dim(\operatorname{rad}(f)) = n$.
(3) *The following statements are equivalent.*
   (a) *f is singular.*

(b) $\operatorname{rad}(f) \neq 0$.
(c) $\operatorname{rank}(f) < n$.
(d) $\det(M) = 0$.

Proof. (1) Let $v \in \operatorname{Null}(M)$. Then $B_f(w, v) = w^t M v = 0$ for all $w \in K^n$, and thus $v \in \operatorname{rad}(f)$. Let $v \in \operatorname{rad}(f)$. Then $B_f(w, v) = 0$ for all $w \in K^n$. Thus $w^t M v = 0$ for all $w \in K^n$, which implies that $Mv = 0$. Therefore, $v \in \operatorname{Null}(M)$.

(2) We have $n = \operatorname{rank}(M) + \dim(\operatorname{Null}(M) = \operatorname{rank}(f) + \dim(\operatorname{rad}(f))$.

(3) The equivalence of the statements follows from the definitions, (1) and (2), and the observation that $\det(M) = 0$ if and only if $\operatorname{Null}(M) \neq 0$. □

Definition 2.5 (Hyperbolic Plane)**.** A quadratic form $f \in K[X_1, X_2]$ is called a hyperbolic plane if f is equivalent to the quadratic form X_1X_2.

The following result is useful to identify a hyperbolic plane.

Lemma 2.6. *Let $f \in K[X_1, X_2]$ be a nonsingular quadratic form. The following statements are equivalent.*

(1) *f has a nontrivial zero over K.*
(2) *$f \cong X_1X_2$.*
(3) *$f \cong \langle 1, -1\rangle$.*

Proof. (1) ⇒ (2): Let $f = aX_1^2 + 2bX_1X_2 + cX_2^2$ with the associated symmetric matrix $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. Assume that f has a nontrivial zero over K. Applying an invertible linear change of variables over K allows us to assume that $f(1, 0) = 0$. Then $a = 0$. Since f is nonsingular, we have $b \neq 0$. Then $f = X_2(2bX_1 + cX_2) \cong X_1X_2$ because X_2 and $2bX_1 + cX_2$ are linearly independent.

(2) ⇒ (3): We have $X_1X_2 \cong (X_1 + X_2)(X_1 - X_2) = X_1^2 - X_2^2$.

(3) ⇒ (1): $X_1^2 - X_2^2$ has a nontrivial zero over K, namely, $X_1 = 1$, $X_2 = 1$. □

For a quadratic form $f \in K[X_1 \ldots, X_n]$, we say that f splits off a hyperbolic plane if f is equivalent to $X_1X_2 + h(X_3, \ldots, X_n)$ for some quadratic form $h \in K[X_3, \ldots, X_n]$. Similarly, we say that f splits off j hyperbolic planes, if

$$f \cong X_1X_2 + X_3X_4 + \cdots + X_{2j-1}X_{2j} + h(X_{2j+1}, \ldots, X_n).$$

Lemma 2.7. *Suppose that $f \in K[X_1, \ldots, X_n]$ is a nonsingular quadratic form that has a nontrivial zero over K. Then $f \cong X_1X_2 + h(X_3, \ldots, X_n)$ for some quadratic form $h \in K[X_3, \ldots, X_n]$.*

Proof. An invertible linear change of variables allows us to assume that $f(1, 0, \ldots, 0) = 0$. Then $f \cong X_1L(X_2, \ldots, X_n) + Q_1(X_2, \ldots, X_n)$ where L is a linear form and Q_1 is a quadratic form, both with coefficients in K. We have $L \neq 0$ because $\operatorname{rad}(f) = (0)$. A second invertible linear change of variables allows us to assume that

$$f \cong X_1X_2 + Q_2(X_2, \ldots, X_n) = X_2(X_1 + c_2X_2 + \cdots + c_nX_n) + h(X_3, \ldots, X_n)$$

where Q_2, h are quadratic forms with coefficients in K. A third invertible linear change of variables allows us to assume that $f \cong X_1X_2 + h(X_3, \ldots, X_n)$. □

3. One Quadratic Form over the Real Numbers

In this section, we prove some basic properties of quadratic forms over $\mathbb{R}$. We begin by introducing some definitions and terminology that are specific to quadratic forms over $\mathbb{R}$. Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ be a quadratic form with associated symmetric matrix M.

Definition 3.1 (Definite, Semi-definite, Indefinite)**.**

(1) We say that f is a definite quadratic form over $\mathbb{R}$ if $f(v)$ has the same sign for every $v \in \mathbb{R}^n\backslash 0$. According to that sign, the quadratic form f is called positive definite or negative definite.
(2) We say that f is a semi-definite quadratic form over $\mathbb{R}$ if $f(v)$ is either non-negative or non-positive for every $v \in \mathbb{R}^n\backslash 0$. If $f(v)$ is non-negative for every $v \in \mathbb{R}^n\backslash 0$, then f is called positive semi-definite. If $f(v)$ is non-positive for every $v \in \mathbb{R}^n\backslash 0$, then f is called negative semi-definite.
(3) We say that f is an indefinite quadratic form over $\mathbb{R}$ if f takes both positive and negative values when evaluated at vectors in $\mathbb{R}^n\backslash 0$.

We say that an $n \times n$ symmetric matrix M is positive definite (negative definite, semi-definite, indefinite) if the quadratic form $f(X_1, \ldots, X_n) = X^t M X$ has that property.

Proposition 3.2. *A quadratic form $f \in \mathbb{R}[X_1, \ldots, X_n]$ has a nontrivial zero over $\mathbb{R}$ if and only if f is not definite.*

Proof. Since f is equivalent to a diagonal form $d_1X_1^2 + \cdots + d_nX_n^2$, it follows that f has a nontrivial zero over $\mathbb{R}$ if and only if $d_1, \ldots, d_n$ do not all have the same sign. □

Definition 3.3 (Signature of a Quadratic Form over $\mathbb{R}$)**.** Suppose that f is equivalent to $\langle d_1, \ldots, d_n\rangle$. Let r be the number of elements in the set $\{d_i \mid d_i > 0, 1 \le i \le n\}$, and s be the number of elements in the set $\{d_i \mid d_i < 0, 1 \le i \le n\}$. The signature of f, denoted by $\operatorname{sgn}(f)$, is defined by $\operatorname{sgn}(f) = r - s$.

Proposition 3.4. *The signature of f does not depend on the diagonalization of f.*

Proof. We can write $\mathbb{R}^n = V_1 \oplus V_2 \oplus \operatorname{rad}(f)$ where $\dim(V_1) = r$ and f is positive definite on V_1, $\dim(V_2) = s$ and f is negative definite on V_2, and $\dim(\operatorname{rad}(f)) = t$ where t is the number of d_i's that equal zero.

Similarly, suppose that f is also equivalent to $\langle d_1', \ldots, d_n'\rangle$ and write $\mathbb{R}^n = V_1' \oplus V_2' \oplus \operatorname{rad}(f)$ where $V_1', V_2', \operatorname{rad}(f)$ have dimensions r', s', t, respectively, as well as the other properties above, and $\operatorname{sgn}(f) = r' - s'$.

Suppose that $r \ne r'$. We can assume that $r < r'$ and then $s > s'$ because $r + s = r' + s' = n - t$. It follows that $(V_2 \oplus \operatorname{rad}(f)) \cap V_1' \ne (0)$ because

$$\dim(V_2 \oplus \operatorname{rad}(f)) + \dim(V_1') = s + t + r' > s + t + r = n.$$

Let $v \in (V_2 \oplus \operatorname{rad}(f)) \cap V_1'$ with $v \ne 0$. Then $f(v) \le 0$ because $v \in V_2 \oplus \operatorname{rad}(f)$, and $f(v) > 0$ because $v \in V_1'$. This contradiction shows that $r = r'$, and thus $s = s'$. Therefore $r - s = r' - s'$. □

Definition 3.5 (Principal Minor)**.** Let M be an $m \times m$ square matrix. A principal sub-matrix of M is a matrix obtained by deleting any k rows and the corresponding k columns. The leading principal sub-matrix of order k of M is obtained by deleting the last $m - k$ rows and columns of M. The determinant of a principal sub-matrix of a matrix M is called a principal minor of M, and the determinant of a leading principal sub-matrix of M is called a leading principal minor of M.

A version of the following result is stated in many textbooks. See for example, [4, p. 328]. We give a particularly nice proof for the convenience of the reader. Note that a principal minor in [4] is what we call a leading principal minor.

Proposition 3.6 (Sylvester's Criterion)**.** *Let A be a real symmetric $n \times n$ matrix. The following statements are equivalent.*

(1) *A is positive definite.*
(2) *Every principal minor of A is positive.*
(3) *Every leading principal minor of A is positive.*

Proof. (1) $\Rightarrow$ (2) Suppose that A is positive definite. Then each principal $i \times i$ submatrix B is positive definite. Since $B = C^tDC$ for some invertible $i \times i$ matrix C and some diagonal $i \times i$ matrix D with positive entries along its main diagonal, it follows that $\det(B) > 0$. Thus (2) holds.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Assume that every leading principal minor of A is positive. The proof is by induction on n with the case $n = 1$ being trivial. Assume that $n \geq 2$ and that the result has been proved for real symmetric $(n-1) \times (n-1)$ matrices. Let

$$A = \begin{pmatrix} M & v \\ v^t & c \end{pmatrix},$$

where M is an $(n-1) \times (n-1)$ matrix, v is an $(n-1) \times 1$ matrix, and $c \in \mathbb{R}$. Since all leading principal minors of A are positive, it follows that every leading principal minor of M is positive. Then $\det(M) > 0$ and so M is invertible, and by induction, M is positive definite. Note that M^{-1} is also symmetric. Let

$$L = \begin{pmatrix} I & 0 \\ v^tM^{-1} & 1 \end{pmatrix}, \quad B = \begin{pmatrix} M & 0 \\ 0 & c - v^tM^{-1}v \end{pmatrix}.$$

Since

$$A = \begin{pmatrix} M & v \\ v^t & c \end{pmatrix} = \begin{pmatrix} I & 0 \\ v^tM^{-1} & 1 \end{pmatrix} \begin{pmatrix} M & 0 \\ 0 & c - v^tM^{-1}v \end{pmatrix} \begin{pmatrix} I & M^{-1}v \\ 0 & 1 \end{pmatrix},$$

we have

$$A = LBL^t.$$

Since $\det(L) = 1$ and $\det(M) > 0$, this gives $\det(A) = \det(L)^2 \det(B) = \det(B) = \det(M)\,(c - v^tM^{-1}v)$. Since $\det(A) > 0$, this gives $c - v^tM^{-1}v > 0$. Therefore B is positive definite. Since $A = LBL^t$, it follows that A is positive definite. ☐

Definition 3.7 (Path-Connected Topological Space)**.** A topological space $\mathbb{X}$ is path-connected if for any $p, q \in \mathbb{X}$, there is a continuous map $\gamma : [0,1] \to \mathbb{X}$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Such a map is called a path from p to q in $\mathbb{X}$.

Definition 3.8 (Unit $m-$sphere, $\mathbb{S}^m$)**.** Let $m \geq 1$ be any natural number. The unit $m-$sphere is defined as

$$\mathbb{S}^m = \Big\{(x_1, \ldots, x_{m+1}) \in \mathbb{R}^{m+1} \Big| \sum_{i=1}^{m+1} x_i^2 = 1\Big\}.$$

Lemma 3.9. *For $m \geq 1$, $\mathbb{S}^m$ is a path-connected subset in $\mathbb{R}^{m+1}$.*

Proof. Let $x = (x_1, \ldots, x_{m+1})$. The map $\sigma : \mathbb{R}^{m+1}\backslash 0 \to \mathbb{S}^m$ given by $\sigma(x) = \dfrac{x}{\sqrt{\sum_{i=1}^{m+1} x_i^2}}$ is a well-defined, continuous map such that $\sigma(\mathbb{R}^{m+1}\backslash 0) = \mathbb{S}^m$. Since $\mathbb{R}^{m+1}\backslash 0$ is path-connected for $m \geq 1$ and a continuous image of a path-connected set is also path-connected, it follows that $\sigma(\mathbb{R}^{m+1}\backslash 0) = \mathbb{S}^m$ is a path-connected subset in $\mathbb{R}^{m+1}$. ☐

The next three propositions determine whether certain subsets in $\mathbb{R}^n$ associated with a quadratic form are path-connected. Some of these results are used later in the proofs of Proposition 4.1 and Proposition 4.4. For the sake of completeness, we give a complete treatment here.

Notation. *Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ be a quadratic form, and let $f_{=0}$ denote the set $\{x \in \mathbb{R}^n\backslash 0 \mid f(x) = 0\}$. The notation $f_{>0}$, $f_{\geq 0}$, $f_{<0}$, $f_{\leq 0}$ are defined in a similar fashion.*

Let $\mathbb{X} \subset \mathbb{R}^n \backslash 0$ be any subset. Consider the relation $\sim$ on $\mathbb{X}$ defined by $p \sim q$ if there exists a path from p to q which lies entirely in $\mathbb{X}$. Then $\sim$ is an equivalence relation on $\mathbb{X}$ and the equivalence classes are called the path-connected components of $\mathbb{X}$. The set $\mathbb{X}$ can be written as a disjoint union of these path-connected components.

To study the path-connected components of the above sets, we will apply an invertible linear change of variables to put the quadratic form into a convenient shape that is easy to work with. Since an invertible linear map of $\mathbb{R}^n$ is a homeomorphism, path-connected components are mapped to path-connected components.

Since a quadratic form defined over $\mathbb{R}$ can be diagonalized, it is easy to check that if a quadratic form $f \in \mathbb{R}[X_1, \ldots, X_n]$ is indefinite and has rank $r \leq n$, then $\pm f$ is equivalent to either $X_1^2 + \cdots + X_{r-1}^2 - X_r^2$ or $X_1^2 + \cdots + X_k^2 - X_{k+1}^2 - \cdots - X_r^2$ with $k \geq 2$, $r - k \geq 2$. We focus on these two quadratic forms in Proposition 3.11 and Proposition 3.12.

Proposition 3.10. *Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ be a quadratic form in $n \geq 2$ variables. Assume that* $\operatorname{rad}(f) \neq 0$. *Then $f_{=0}$, $f_{\leq 0}$, and $f_{\geq 0}$ are path-connected.*

Proof. Let $\operatorname{rank}(f) = m$. Then we can assume that $f \in \mathbb{R}[X_1, \ldots, X_m]$, and $m < n$ because $\operatorname{rad}(f) \neq 0$. Let $u = (a_1, \ldots, a_m, \ldots, a_n) \in f_{=0}$. Then $u = (a_1, \ldots, a_n)$ is path-connected to $u' = (a_1, \ldots, a_m, 1, a_{m+2}, \ldots, a_n)$ by a line segment that lies in $f_{=0}$, and u' is path-connected to e_{m+1} by a line segment that lies in $f_{=0}$. Therefore, u is path-connected to e_{m+1} in $f_{=0}$. Since any two points in $f_{=0}$ are path-connected to e_{m+1}, it follows that $f_{=0}$ is path-connected.

The proof that $f_{\leq 0}$ and $f_{\geq 0}$ are path-connected is obtained by replacing each $f_{=0}$ with either $f_{\leq 0}$ or $f_{\geq 0}$. $\square$

Proposition 3.11. *Let $f = X_1^2 + \cdots + X_{r-1}^2 - X_r^2 \in \mathbb{R}[X_1, \ldots, X_n]$, $r \leq n$.*

(1) (a) *If $2 = r = n$, then $f_{>0}$, $f_{\geq 0}$, $f_{<0}$, and $f_{\leq 0}$ each have two path-connected components $\mathcal{B}_1$ and $\mathcal{B}_2$ where $\mathcal{B}_2 = (-1)\mathcal{B}_1$.*
(b) *If $2 = r < n$, then $f_{\geq 0}$ and $f_{\leq 0}$ are path-connected.*
(c) *If $2 = r < n$, then $f_{>0}$ and $f_{<0}$ each have two path-connected components $\mathcal{B}_1$ and $\mathcal{B}_2$ where $\mathcal{B}_2 = (-1)\mathcal{B}_1$.*

(2) (a) *If $3 \leq r < n$, then $f_{\leq 0}$ is path-connected.*
(b) *If $3 \leq r$, then $f_{>0}$ and $f_{\geq 0}$ are path-connected.*
(c) *If $3 \leq r = n$, then $f_{\leq 0}$ is not path-connected.*
(d) *If $3 \leq r$ then $f_{<0}$ is not path-connected.*

Let $u = (a_1, \ldots, a_n), v = (b_1, \ldots, b_n) \in f_{\leq 0}\,(f_{<0})$. In (2c) and (2d), u, v are path-connected in $f_{\leq 0}\,(f_{<0})$ if and only if a_r, b_r have the same sign. In particular, $f_{\leq 0}\,(f_{<0})$ has two path-connected components $\mathcal{B}_1$ and $\mathcal{B}_2$ where $\mathcal{B}_2 = (-1)\mathcal{B}_1$.

(3) (a) *If $2 \leq r < n$, then $f_{=0}$ is path-connected.*
(b) *Let $2 \leq r = n$.*
(i) *If $r = 2$, then $f_{=0}$ has four path-connected components.*
(ii) *Assume $3 \leq r = n$. Let $u = (a_1, \ldots, a_r), v = (b_1, \ldots, b_r) \in f_{=0}$. Then u, v are path-connected in $f_{=0}$ if and only if a_r, b_r have the same sign. In particular, $f_{=0}$ has two path-connected components $\mathcal{B}_1$ and $\mathcal{B}_2$ where $\mathcal{B}_2 = (-1)\mathcal{B}_1$.*

Proof. (1) Note that in this case $f = X_1^2 - X_2^2$.

(a) If $2 = r = n$, then $(1, 0)$ and $(-1, 0)$ lie in $f_{\geq 0}$ but are not path-connected in $f_{\geq 0}$ because a path joining $(1, 0)$ and $(-1, 0)$ would contain a point $(0, b)$ for some nonzero $b \in \mathbb{R}$, but such a point would not lie in $f_{\geq 0}$. Similarly,

	$f_{>0}$	$f_{<0}$	$f_{=0}$	$f_{\geq 0}$	$f_{\leq 0}$
$r = 2 = n$	No†	No†	No	No†	No †
$r = 2,\ r < n$	No †	No†	Yes	Yes	Yes
$r \geq 3,\ r = n$	Yes	No†	No†	Yes	No†
$r \geq 3,\ r < n$	Yes	No†	Yes	Yes	Yes

TABLE 1. Summary of the results from Proposition 3.11. The sets marked with "No†" have two path-connected components $\mathcal{B}_1$ and $\mathcal{B}_2$ where $\mathcal{B}_2 = (-1)\mathcal{B}_1$.

$f_{>0}$ is also not path-connected. Since $r = 2$, by replacing f with $-f$, we get that $f_{<0}$ and $f_{\leq 0}$ are each not path-connected.

(b) If $2 = r < n$, then $f_{\geq 0}$ and $f_{\leq 0}$ are path-connected by Proposition 3.10.

(c) The points $(1, 0, \ldots, 0)$ and $(-1, 0, \ldots, 0)$ lie in $f_{>0}$ but are not path-connected in $f_{>0}$ because a path joining $(1, 0, \ldots, 0)$ and $(-1, 0, \ldots, 0)$ would contain a point $(0, b_2, \ldots, b_n)$ where some $b_i \neq 0$, $2 \leq i \leq n$, but such a point lies in $f_{\leq 0}$. Since $r = 2$, by replacing f with $-f$, we get that $f_{<0}$ is also not path-connected.

In (a) and (c), it follows that the indicated set is the union of two path-connected components $\mathcal{B}_1$ and $\mathcal{B}_2$ where $\mathcal{B}_2 = (-1)\mathcal{B}_1$.

(2) (a) If $3 \leq r < n$, then $f_{\leq 0}$ is path-connected by Proposition 3.10.

(b) Let $3 \leq r$ and let $u = (a_1, \ldots, a_n)$, $v = (b_1, \ldots, b_n) \in f_{>0}$. Then $a_i \neq 0$ and $b_j \neq 0$ for some i, j where $1 \leq i,\ j \leq r - 1$. Note that $u = (a_1, \ldots, a_n)$ is path-connected to $u' = (a_1, \ldots, a_{r-1}, 0, \ldots, 0)$ by a line segment that lies in $f_{>0}$, and $v = (b_1, \ldots, b_n)$ is path-connected to $v' = (b_1, \ldots, b_{r-1}, 0, \ldots, 0)$ by a line segment that lies in $f_{>0}$. Since $r \geq 3$, we have that $\mathbb{R}^{r-1} \backslash \{0\}$ is path-connected and thus u', v' are path-connected by a line segment that lies in $f_{>0}$. Therefore, $f_{>0}$ is path-connected.

If $3 \leq r < n$, then $f_{\geq 0}$ is path-connected by Proposition 3.10.

We can now assume that $3 \leq r = n$. Let $u = (a_1, \ldots, a_n)$, $v = (b_1, \ldots, b_n) \in f_{\geq 0}$. Since $r = n$, we have $a_i, b_j \neq 0$ for some i, j where $1 \leq i,\ j \leq r - 1$. Note that $u = (a_1, \ldots, a_n)$ is path-connected to $u' = (a_1, \ldots, a_{r-1}, 0)$ by a line segment that lies in $f_{\geq 0}$, and $v = (b_1, \ldots, b_n)$ is path-connected to $v' = (b_1, \ldots, b_{r-1}, 0)$ by a line segment that lies in $f_{\geq 0}$.

Since $r \geq 3$, we have that $\mathbb{R}^{r-1} \backslash \{0\}$ is path-connected and thus u', v' are path-connected by a line segment that lies in $f_{\geq 0}$. Therefore, $f_{\geq 0}$ is path-connected.

(c) Let $u = (a_1, \ldots, a_n), v = (b_1, \ldots, b_n) \in f_{\leq 0}$. Then $a_r \neq 0$ and $b_r \neq 0$ because $r = n$. First assume that a_r, b_r have opposite signs. A path from u to v must pass through some point $w = (c_1, \ldots, c_{r-1}, 0)$ where $f(w) > 0$. Thus u and v are not path-connected in $f_{\leq 0}$.

Now suppose that a_r, b_r have the same signs. Note that u and $u' = (0, \ldots, 0, a_r)$ are path-connected by a line segment that lies in $f_{\leq 0}$, and v and $v' = (0, \ldots, 0, b_r)$ are path-connected by a line segment that lies in $f_{\leq 0}$. Since a_r, b_r have the same signs, it follows that u', v' are path-connected by a line segment that lies in $f_{\leq 0}$. Therefore, u, v are path-connected in $f_{\leq 0}$. Since $u \in f_{\leq 0}$ if and only if $-u \in f_{\leq 0}$, it follows that $f_{\leq 0}$ is the union of two path-connected components $\mathcal{B}_1$ and $\mathcal{B}_2$ where $\mathcal{B}_2 = (-1)\mathcal{B}_1$.

(d) Let $u = (a_1, \ldots, a_n), v = (b_1, \ldots, b_n) \in f_{<0}$. Then $a_r \neq 0$ and $b_r \neq 0$. First assume that a_r, b_r have opposite signs. A path from u to v must pass through some point $w = (c_1, \ldots, c_{r-1}, 0, c_{r+1}, \ldots, c_n)$ where $f(w) \geq 0$. Thus u and v are not path-connected in $f_{<0}$.

Now suppose that a_r, b_r have the same signs. Note that u and $u' = (0, \ldots, 0, a_r, 0, \ldots, 0)$ are path-connected by a line segment that lies in $f_{<0}$, and v and $v' = (0, \ldots, 0, b_r, 0, \ldots, 0)$ are path-connected by a line segment that lies in $f_{<0}$. Since u' is path-connected to v' by a line segment that lies in $f_{<0}$, it follows that u, v are path-connected in $f_{<0}$. Since $u \in f_{<0}$ if and only if $-u \in f_{<0}$, it follows that $f_{<0}$ is the union of two path-connected components $\mathcal{B}_1$ and $\mathcal{B}_2$ where $\mathcal{B}_2 = (-1)\mathcal{B}_1$.

(3) (a) This case follows from Proposition 3.10.

(b) First, assume $2 = r = n$. Then $f = X_1^2 - X_2^2$. The four path-connected components are given by $X_2 = \pm X_1,\ X_1 > 0$, and $X_2 = \pm X_1,\ X_1 < 0$.

Now assume that $3 \leq r = n$. Let $u = (a_1, \ldots, a_r)$, $v = (b_1, \ldots, b_r) \in f_{=0}$. Then $a_r \neq 0$, $b_r \neq 0$, and $a_i \neq 0$, $b_j \neq 0$ for some $1 \leq i, j \leq r-1$. Suppose that a_r and b_r have opposite signs. Then a path from u to v in $f_{=0}$ would pass through a nonzero vector of the form $w = (c_1, \ldots, c_{r-1}, 0)$ where $f(w) > 0$. Therefore, u and v are not path-connected in $f_{=0}$.

Now suppose that a_r and b_r have the same signs. There exist $c, d \in \mathbb{R}_{>0}$ such that $u' = c(a_1, \ldots, a_{r-1}), v' = d(b_1, \ldots, b_{r-1}) \in \mathbb{S}^{r-2}$. Then $u'' = \left(u', \dfrac{a_r}{|a_r|}\right)$ and $v'' = \left(v', \dfrac{b_r}{|b_r|}\right)$ lie in $f_{=0}$. Note that $\dfrac{a_r}{|a_r|} = \dfrac{b_r}{|b_r|} = \pm 1$.

We have that u is path-connected to u'' by a line segment that lies in $f_{=0}$ and v is path-connected to v'' by a line segment that lies in $f_{=0}$. Next, u'' is path-connected to v'' by a path that lies in $f_{=0}$ by Lemma 3.9 because either $u'', v'' \in \mathbb{S}^{r-2} \times \{1\}$ or $u'', v'' \in \mathbb{S}^{r-2} \times \{-1\}$. Thus u is path-connected to v in $f_{=0}$.

Therefore, u is path-connected to v in $f_{=0}$ if and only if a_r, b_r have the same sign. It follows that $f_{=0}$ is the union of two path-connected components $\mathcal{B}_1$ and $\mathcal{B}_2$ where $\mathcal{B}_2 = (-1)\mathcal{B}_1$.

□

Proposition 3.12. *Let* $f = X_1^2 + \cdots + X_k^2 - X_{k+1}^2 - \cdots - X_r^2 \in \mathbb{R}^n[X_1, \ldots X_n]$ *such that* $\operatorname{rank}(f) = r \leq n$. *Assume that* $k \geq 2,\ r - k \geq 2$. *Then* $f_{>0}$, $f_{\geq 0}$, $f_{<0}$, $f_{\leq 0}$, *and* $f_{=0}$ *are each path-connected.*

Proof. First, we show that $f_{>0}$ is path-connected. Let $u = (a_1, \ldots, a_n)$, $v = (b_1, \ldots, b_n)$ be points in $f_{>0}$. This implies that $a_i \neq 0, b_j \neq 0$ for some $1 \leq i, j \leq k$. There exist $c, d \in \mathbb{R}_{>0}$ such that letting

$$u' = (ca_1, \ldots, ca_k, 0, \ldots, 0),\ v' = (db_1, \ldots, db_k, 0, \ldots, 0),$$

we have $u', v' \in \mathbb{S}^{k-1} \times \{0\}^{n-k} \subset f_{>0}$, where $\mathbb{S}^{k-1}$ is the unit sphere in $\mathbb{R}^k$.

Since $\mathbb{S}^{k-1}$ is path-connected when $k \geq 2$, u' and v' are path-connected in $f_{>0}$. Since u is path-connected to u' by a line segment that lies in $f_{>0}$, and v' is path-connected to v by a line segment that lies in $f_{>0}$, it follows that u is path-connected to v in $f_{>0}$ and hence $f_{>0}$ is path-connected. It follows that $f_{<0} = (-f)_{>0}$ is also path-connected.

If $r < n$, then $\operatorname{rad}(f) \neq 0$ and hence $f_{\geq 0}$, $f_{\leq 0}$, and $f_{=0}$ are each path-connected by Proposition 3.10.

Now assume that $r = n$. We next show that $f_{=0}$ is path-connected. Let u, v be points in $f_{=0}$. Write $u = (p, q)$ and $v = (r, s)$ where $p, r \in \mathbb{R}^k \backslash 0$ and $q, s \in \mathbb{R}^{n-k} \backslash 0$. There exist $c, d \in \mathbb{R}_{>0}$ such that

$$(cp, cq), (dr, ds) \in \mathbb{S}^{k-1} \times \mathbb{S}^{n-k-1} \subset f_{=0}.$$

Note that u is path-connected to (cp, cq) by a line segment that lies in $f_{=0}$, and v is path-connected to (dr, ds) by a line segment that lies in $f_{=0}$. Since $k \geq 2$ and $n-k \geq 2$, it follows that $\mathbb{S}^{k-1} \times \mathbb{S}^{n-k-1}$ is path-connected. Thus (cp, cq) is path-connected to (dr, ds) in $\mathbb{S}^{k-1} \times \mathbb{S}^{n-k-1}$. It follows that u is path-connected to v in $f_{=0}$, and thus $f_{=0}$ is path-connected.

We now show that $f_{\geq 0}$ is path-connected. We have $e_1 \in f_{>0}$ and $e_1 + e_{k+1} \in f_{=0}$. The line segment joining e_1 and $e_1 + e_{k+1}$ lies in $f_{\geq 0}$. Since $f_{>0}$ and $f_{=0}$ are each path-connected and there is a path in $f_{\geq 0}$ joining e_1 and $e_1 + e_{k+1}$, it follows that $f_{>0} \cup f_{=0} = f_{\geq 0}$ is path-connected.

It follows that $f_{\leq 0} = (-f)_{\geq 0}$ is also path-connected. □

The parts of Proposition 3.11 that are needed for the proofs of Proposition 4.1 and Proposition 4.4 are parts 1(a), 1(c), 2(b), 2(d), and 3.

4. Two Quadratic Forms over the Real Numbers

Our proof of Proposition 4.4 below is slightly different from the proof given in [7, Lemma 1 (i)]. We first give Swinnerton-Dyer's proof of [7, Lemma 1 (ii)] in Proposition 4.1 and then use this result to give a simpler proof of [7, Lemma 1 (i)] in Proposition 4.4. We have added many details not included in Swinnerton-Dyer's exposition.

Proposition 4.1. *Let $f, g \in \mathbb{R}[X_1, \ldots, X_n]$ be quadratic forms with $n \geq 2$ and assume that f is indefinite. Then there exist real zeros v, w on $f_{=0}$ such that $g(v) > 0$ and $g(w) < 0$ if and only if $\lambda f + g$ is indefinite for all $\lambda \in \mathbb{R}$.*

Proof. If there exist real zeros v, w on $f_{=0}$ such that $g(v) > 0$ and $g(w) < 0$, then $\lambda f + g$ is indefinite for all $\lambda \in \mathbb{R}$ because $(\lambda f + g)(v) > 0$ and $(\lambda f + g)(w) < 0$.

Now assume that $\lambda f + g$ is indefinite for all $\lambda \in \mathbb{R}$. Suppose that $g \geq 0$ on $f_{=0}$. We will obtain a contradiction. The case $g \leq 0$ on $f_{=0}$ is handled by replacing g with $-g$ and noting that $-g \geq 0$ on $f_{=0}$.

The set $(\lambda f + g)_{<0}$ does not meet $f_{=0}$ for any real λ because $f_{=0}$ lies entirely in $g \geq 0$. For any $\lambda \in \mathbb{R}$, $(\lambda f + g)_{<0}$ is a non-empty, open set. Since $\lambda f + g$ is indefinite, its rank is at least 2. Proposition 3.11, Proposition 3.12, and the comments above Proposition 3.10 imply that the set $(\lambda f + g)_{<0}$ is either path-connected or has two path-connected components $\mathcal{B}_1$, $\mathcal{B}_2$ where $\mathcal{B}_2 = (-1)\mathcal{B}_1$. Since $f_{>0}$ and $f_{<0}$ are disjoint open sets, it follows that each path-connected component lies entirely in either $f_{>0}$ or $f_{<0}$. Since f is a quadratic form, $f(v) = f(-v)$ for all $v \in \mathbb{R}^n$, so it follows that if $(\lambda f + g)_{<0}$ has two path-connected components, then both path components lie entirely in either $f_{>0}$ or $f_{<0}$. Thus $(\lambda f + g)_{<0}$ lies entirely in either $f_{>0}$ or $f_{<0}$.

Define

$$\Lambda_1 = \left\{ \lambda \in \mathbb{R} \;\middle|\; \lambda f + g < 0 \text{ lies in } f > 0 \right\},$$

and

$$\Lambda_2 = \left\{ \lambda \in \mathbb{R} \;\middle|\; \lambda f + g < 0 \text{ lies in } f < 0 \right\}.$$

Then Λ_1 and Λ_2 are disjoint and $\Lambda_1 \cup \Lambda_2 = \mathbb{R}$.

Claim. *Λ_1 and Λ_2 are non-empty subsets of $\mathbb{R}$.*

Since f is indefinite, there exist $v, u \in \mathbb{R}^n \backslash 0$ such that $f(v) > 0$ and $f(u) < 0$. Choose $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 < \frac{-g(v)}{f(v)}$ and $\lambda_2 > \frac{-g(u)}{f(u)}$. Then $\lambda_1 f(v) + g(v) < 0$ and so $\lambda_1 \in \Lambda_1$. Similarly, $\lambda_2 f(u) + g(u) < 0$, and so $\lambda_2 \in \Lambda_2$.

Claim. *Λ_1 and Λ_2 are open sets in $\mathbb{R}$.*

Let $\lambda \in \Lambda_1$. Then there exists $v \in \mathbb{R}^n$ such that $\lambda f(v) + g(v) < 0$ and $f(v) > 0$. This implies that $\lambda < \frac{-g(v)}{f(v)}$. If $\lambda' < \frac{-g(v)}{f(v)}$, then $\lambda' f(v) + g(v) < 0$. Hence, $\lambda \in \left(-\infty, \dfrac{-g(v)}{f(v)}\right) \subseteq \Lambda_1$.

Similarly, for $\lambda \in \Lambda_2$ there exists $u \in \mathbb{R}^n$ such that $\lambda f(u) + g(u) < 0$ and $f(u) < 0$. This implies that $\lambda > \frac{-g(u)}{f(u)}$. If $\lambda' > \frac{-g(u)}{f(u)}$, then $\lambda' f(u) + g(u) < 0$. Hence, $\lambda \in \left(\dfrac{-g(u)}{f(u)}, \infty\right) \subseteq \Lambda_2$. This proves the claim.

The previous two claims show that $\mathbb{R}$ can be written as a disjoint union of two non-empty open sets, which is a contradiction. Therefore, there exist real zeros on $f_{=0}$ that give either sign to g. $\square$

Remark 4.2. In Proposition 4.1, suppose that f is a positive semi-definite, but not a definite quadratic form. If there exist real zeros on $f_{=0}$ that give either sign to g, then $\lambda f + g$ is indefinite for all $\lambda \in \mathbb{R}$. However, the converse statement is not true, as the next example shows.

Example 4.3. Let $f = X_1^2 + X_2^2$ and $g = X_1X_3 + X_2X_4$ be quadratic forms in $\mathbb{R}[X_1, X_2, X_3, X_4]$. Then f is positive semi-definite and g is indefinite. For any nonzero $\lambda \in \mathbb{R}$,

$$\lambda f + g = \lambda X_1^2 + \lambda X_2^2 + X_1X_3 + X_2X_4$$

is indefinite because

$$(\lambda f + g)(1, 1, 0, 0) = 2\lambda \text{ and } (\lambda f + g)(1, 1, -2\lambda, -2\lambda) = -2\lambda$$

are opposite in signs when $\lambda \neq 0$. Therefore, $\lambda f + g$ is indefinite for all $\lambda \in \mathbb{R}$, but $g|_{f=0} = 0$.

Proposition 4.4. *Let $f, g \in \mathbb{R}[X_1, \ldots, X_n]$ be quadratic forms with $n \geq 3$. Then the following statements are equivalent.*

(1) *The set $f = g = 0$ contains a nontrivial real zero.*
(2) *$\lambda f + \mu g$ is not definite for any real λ, μ, not both zero.*
(3) *For every real λ, μ, not both zero, $\lambda f + \mu g$ has a nontrivial real zero.*

Proof. A definite quadratic form has no nontrivial real zero. A quadratic form that is not definite is either semi-definite or indefinite, and in each case, the quadratic form has a nontrivial real zero. Thus (2) and (3) are equivalent.

(1) $\Rightarrow$ (2). If $f = g = 0$ has a nontrivial real zero, then $\lambda f + \mu g$ is not definite for any real λ, μ, not both zero.

(2) $\Rightarrow$ (1). Suppose that $\lambda f + \mu g$ is not definite for any real λ, μ not both zero. We have the following two cases:

Case 1. Suppose there exists a semi-definite form in $\mathcal{P}_{\mathbb{R}}(f, g)$. Without loss of generality, we may assume that f is a positive semi-definite quadratic form. Since f is not definite, after an invertible linear transformation, we may assume that

$$f(X_1, \ldots, X_n) = X_1^2 + \cdots + X_r^2,$$

where $r < n$ is the rank of f, and

$$g = \sum_{1 \leq i \leq j \leq n} a_{ij} X_i X_j.$$

Suppose first that $g(0, \ldots, 0, X_{r+1}, \ldots, X_n)$ has no nontrivial zero over $\mathbb{R}$. Then $g(0, \ldots, 0, X_{r+1}, \ldots, X_n)$ is definite, and by replacing g with $-g$ if necessary, we can assume that $g(0, \ldots, 0, X_{r+1}, \ldots, X_n)$ is positive definite.

For $\lambda \in \mathbb{R}$, consider the symmetric matrix corresponding to $\lambda f + g$:

$$\begin{array}{cc} & \begin{array}{cc} \qquad\qquad r \qquad\qquad & n-r \end{array} \\ \begin{array}{c} r \\ \\ n-r \end{array} & \left(\begin{array}{ccc|c} \lambda+\alpha_{11} & & * & \\ & \ddots & & * \\ * & & \lambda+\alpha_{rr} & \\ \hline & * & & * \end{array}\right) \end{array}$$

Since $g(0,\ldots,0,X_{r+1},\ldots,X_n)$ is positive definite, the $n-r$ leading principal minors starting from the lower right corner of the above matrix are positive by Proposition 3.6. Since λ appears only in the diagonal entries, we can choose λ_0 large enough so that all leading principal minors starting from the lower right corner are positive. Hence, Proposition 3.6 implies that $\lambda_0 f + g$ is a positive definite quadratic form, which is a contradiction.

Thus $g(0,\ldots,0,X_{r+1},\ldots,X_n)$ has a nontrivial zero over $\mathbb{R}$. This zero is then a nontrivial common zero over $\mathbb{R}$ of f, g.

Case 2. Assume that every form in $\mathcal{P}_{\mathbb{R}}(f,g)$ is indefinite. Since f is indefinite and $n \geq 3$, Proposition 3.11, Proposition 3.12, and the comments above Proposition 3.10 imply that $f_{=0}$ is either path-connected or is a union of two path-connected components of the form $\mathcal{B}_1 \cup \mathcal{B}_2$ where $\mathcal{B}_2 = -\mathcal{B}_1$.

Proposition 4.1 implies that there exist $u, v \in f_{=0}$ such that $g(u) > 0$ and $g(v) < 0$. We can assume that u, v lie in the same path-connected component $\mathcal{B}$ of $f_{=0}$ because v can be replaced by $-v$ if necessary.

Consider a path $\gamma : [0,1] \to \mathcal{B}$ where γ is a continuous function satisfying $\gamma(0) = u$ and $\gamma(1) = v$. Since $g : \mathcal{B} \to \mathbb{R}$ is continuous, we have $g \circ \gamma : [0,1] \to \mathbb{R}$ is a continuous function. The image of $g \circ \gamma$ is connected because the continuous image of a connected space is connected. Since $g \circ \gamma(0) = g(u) > 0$ and $g \circ \gamma(1) = g(v) < 0$, it follows that there exists $c \in (0,1)$ such that $g \circ \gamma(c) = 0$. Thus $\gamma(c) \in \mathcal{B} \cap g_{=0} \subseteq f_{=0} \cap g_{=0}$.

□

Example 4.5. Let $f = 2X_1X_2$, $g = X_1^2 - X_2^2$. The pair f, g has no nontrivial zeros over $\mathbb{R}$, or even over $\mathbb{C}$. Every form in the $\mathbb{R}$-pencil is indefinite. This shows that the hypothesis $n \geq 3$ in Proposition 4.4 is necessary.

Example 4.6. This example shows that condition (2) in Proposition 4.4 is not the same as the condition that $\lambda f + \mu g$ is indefinite for every real λ, μ, not both zero. In particular, it really is necessary to consider Case 1 in the proof that (2) $\Rightarrow$ (1). Let $1 \leq j \leq n-2$ and let

$$f = X_1^2 + \cdots + X_j^2$$

$$g = h(X_1,\ldots,X_j) + X_{j+1}^2 + \cdots + X_{n-1}^2 - X_n^2,$$

where h is any quadratic form with real coefficients. Then f is positive semi-definite, but not definite, and $\lambda f + \mu g$ is indefinite for every real λ, μ with $\mu \neq 0$. Also, $(0,\ldots,0,1,1)$ is a real nontrivial common zero of f, g.

Remark 4.7. The proof of Proposition 4.4 is motivated by [7, Lemma 1 (i)]. However, the proof given in [7, Lemma 1 (i)] did not consider Case 1 in the proof of (2) $\Rightarrow$ (1). The argument given in the proof of [7, Lemma 1 (i)] fails when f is semi-definite.

5. Signature of a Quadratic Form

Let $f, g \in \mathbb{R}[X_1, \ldots, X_n]$ be two quadratic forms and let M_f, M_g denote the symmetric matrices corresponding to f, g, respectively. Let

$$D(\lambda, \mu) = \det(\lambda f + \mu g) = \det(\lambda M_f + \mu M_g),$$

and call this the determinant polynomial of f, g. Either $D(\lambda, \mu) = 0$ or $D(\lambda, \mu)$ is a nonzero homogeneous form of degree n in the variables λ, μ. See Example 6.6 below for an example where $D(\lambda, \mu) = 0$. Let

$$T = \{(\lambda, \mu) \in \mathbb{S}^1 \subset \mathbb{R}^2 \mid \det(\lambda f + \mu g) = 0\}.$$

Lemma 5.1. *Assume that $D(\lambda, \mu) = \det(\lambda f + \mu g)$ is a nonzero homogeneous form in the variables λ, μ. Then $|T| \le 2n$.*

Proof. The hypothesis implies that $D(\lambda, \mu) = \det(\lambda f + \mu g)$ is a nonzero homogeneous form of degree n in the variables λ, μ. Since $D(\lambda, \mu)$ has at most n distinct linear factors defined over $\mathbb{R}$, the equation $D(\lambda, \mu) = 0$ has at most $2n$ distinct zeros on $\mathbb{S}^1$ because each linear factor $a\lambda + b\mu$ gives exactly two zeros $\left(\frac{-b}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}\right)$ and $\left(\frac{b}{\sqrt{a^2+b^2}}, \frac{-a}{\sqrt{a^2+b^2}}\right)$ of $D(\lambda, \mu)$ on $\mathbb{S}^1$. Therefore, $|T| \le 2n$. □

Next, we define the signature map

$$\begin{aligned} \mathrm{Sgn} : \mathbb{S}^1 &\to \mathbb{Z} \\ (\lambda, \mu) &\mapsto \mathrm{sgn}(\lambda f + \mu g) \end{aligned}$$

For any $n \times n$ matrix M and integer k with $1 \le k \le n$, let $M^{(k)}$ denote the upper left $k \times k$ sub-matrix of M, and let $d_k = \det(M^{(k)})$.

In the following, we give the discrete topology to $\mathbb{Z}$, which is the same as the subspace topology inherited from the standard topology on $\mathbb{R}$.

Proposition 5.2. *Assume that $D(\lambda, \mu)$ is nonzero. The signature map* Sgn *is constant on each connected component of $\mathbb{S}^1 - T$ and thus* Sgn *is continuous at all points of $\mathbb{S}^1$ except for the finitely many points that lie in $T \subset \mathbb{S}^1$.*

Proof. The set T is finite by Lemma 5.1. Let $(\lambda_0, \mu_0) \in \mathbb{S}^1 - T$. Then $\lambda_0 f + \mu_0 g$ is a nonsingular quadratic form. Since $\mathrm{rad}(\lambda_0 f + \mu_0 g) = 0$, we can write $\mathbb{R}^n = V \oplus W$ where $(\lambda_0 f + \mu_0 g)(v) > 0$ for all nonzero $v \in V$, $(\lambda_0 f + \mu_0 g)(w) < 0$ for all nonzero $w \in W$, $\dim(V) = r$, $\dim(W) = s$, $\mathrm{sgn}(\lambda_0 f + \mu_0 g) = r - s$. The subspaces V, W are not uniquely determined, but $\dim(V), \dim(W)$ are uniquely determined by Proposition 3.4 and its proof. Let $\{v_1, \ldots, v_r\}$ be a basis of V and $\{v_{r+1}, \ldots, v_{r+s}\}$ a basis of W. Let

$$S_V = \{a_1 v_1 + \cdots + a_r v_r \in V \mid a_1^2 + \cdots + a_r^2 = 1\},$$

$$S_W = \{a_{r+1} v_{r+1} + \cdots + a_n v_n \in W \mid a_{r+1}^2 + \cdots + a_n^2 = 1\}.$$

We take S_V to be the empty set if $r = 0$, and similarly for S_W if $s = 0$. The sets S_V, S_W are compact subsets of $\mathbb{R}^r$, $\mathbb{R}^s$, respectively.

The function $\tau_V : S_V \times \mathbb{S}^1 \to \mathbb{R}$ defined by $\tau_V(v, \lambda, \mu) = v^t M_{\lambda f + \mu g} v - v^t M_{\lambda_0 f + \mu_0 g} v$ is a polynomial function of the entries of v and of λ, μ, and thus a continuous function, and similarly for the corresponding function $\tau_W : S_W \times \mathbb{S}^1 \to \mathbb{R}$. Since $S_V \times \mathbb{S}^1$ and $S_W \times \mathbb{S}^1$ are compact subsets of the metric spaces $\mathbb{R}^r \times \mathbb{R}^2$, $\mathbb{R}^s \times \mathbb{R}^2$, respectively, uniform continuity implies that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if (λ, μ) lies in U_δ, the open neighborhood around (λ_0, μ_0) of radius δ, then

$$|v^t M_{\lambda f + \mu g} v - v^t M_{\lambda_0 f + \mu_0 g} v| < \varepsilon,$$

for every $v \in S_V$, with a similar statement holding for every $v \in S_W$.

If S_V is nonempty, let

$$A_V = \min_{v \in S_V} \{|v^t M_{\lambda_0 f + \mu_0 g} v|\},$$

with A_W defined similarly if S_W is nonempty. Note that if S_V is nonempty, then $A_V > 0$ because S_V is compact, and similarly, $A_W > 0$ if S_W is nonempty.

Let $\varepsilon = \min\{A_V, A_W\}$ if S_V and S_W are both nonempty. Otherwise, let $\varepsilon = A_V$ if only S_V is nonempty, and let $\varepsilon = A_W$ if only S_W is nonempty.

Since $|v^t M_{\lambda_0 f+\mu_0 g} v| \geq \varepsilon$ for all $v \in S_V \cup S_W$, and since for all $(\lambda, \mu) \in U_\delta$, we have $|v^t M_{\lambda f+\mu g} v - v^t M_{\lambda_0 f+\mu_0 g} v| < \varepsilon$, it follows that $v^t M_{\lambda_0 f+\mu_0 g} v$ and $v^t M_{\lambda f+\mu g} v$ have the same sign for all $v \in S_V$ and $(\lambda, \mu) \in U_\delta$, with a similar statement for all $v \in S_W$.

Therefore for all $(\lambda, \mu) \in U_\delta$, we have $(\lambda f + \mu g)(v) > 0$ for all $v \in S_V$ and thus also for all $v \in V$, and $(\lambda f + \mu g)(v) < 0$ for all $v \in S_W$ and thus also for all $v \in W$. It follows that the decomposition $\mathbb{R}^n = V \oplus W$ can be used to compute both $\operatorname{sgn}(\lambda_0 f + \mu_0 g)$ and $\operatorname{sgn}(\lambda f + \mu g)$ and this gives

$$\operatorname{Sgn}(\lambda_0, \mu_0) = \dim(V) - \dim(W) = \operatorname{Sgn}(\lambda, \mu),$$

for all $(\lambda, \mu) \in U_\delta$. This shows that $\operatorname{Sgn} : \mathbb{S}^1 - T \to \mathbb{Z}$ is a locally constant function. Since $\mathbb{Z}$ is given the discrete topology it follows that Sgn is continuous on all the points in $\mathbb{S}^1 - T$. If $\mathcal{C}_i$ is a connected component of $\mathbb{S}^1 - T$, then $\operatorname{Sgn}(\mathcal{C}_i)$ is also connected. Since the only connected sets in $\mathbb{Z}$ are singleton sets, we see that $\operatorname{Sgn}(\mathcal{C}_i)$ is a constant. □

Proposition 5.3. *Assume that $D(\lambda, \mu) \neq 0$. For $(\lambda, \mu) \in \mathbb{S}^1$, the signature of the quadratic form $\lambda f + \mu g$ changes only as we pass through a point T on $\mathbb{S}^1$ and it changes by at most twice the dimension of the radical of the form.*

Proof. The proof of the first part of this proposition follows from Proposition 5.2. We now show that as we pass through a point (λ_0, μ_0) in T on $\mathbb{S}^1_{\mu>0}$, the signature changes by at most twice the dimension of the radical of the form $\lambda_0 f + \mu_0 g$. Let $\operatorname{rank}(\lambda_0 f + \mu_0 g) = r < n$. Without loss of generality, we may assume that $\lambda_0 f + \mu_0 g \in \mathbb{R}[X_1, \ldots, X_r]$. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s$ denote the connected components of $\mathbb{S}^1 - T$. Proposition 5.2 implies that Sgn is constant on each $\mathcal{C}_i$. Let $\mathcal{C}_1, \mathcal{C}_2$ be the two consecutive components such that (λ_0, μ_0) is the point of singularity that disconnects $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathbb{S}^1$. The form $\lambda f + \mu g$ is nonsingular for all $(\lambda, \mu) \in \mathcal{C}_1 \cup \mathcal{C}_2$. For all $(\lambda, \mu) \in \{\mathcal{C}_1 \cup \mathcal{C}_2 \cup (\lambda_0, \mu_0)\}$, $\lambda_0 f + \mu_0 g$ and $\left(\lambda f + \mu g\big|_{X_{r+1}=\cdots=X_n=0}\right)$ are quadratic forms in r variables, and in this case $\lambda_0 f + \mu_0 g$ is nonsingular when considered as a form in r variables. We define the following map which is the restriction of Sgn defined above.

$$\begin{aligned} \operatorname{Sgn}_1 : \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{(\lambda_0, \mu_0)\} &\to \mathbb{Z} \\ (\lambda, \mu) &\mapsto \operatorname{sgn}\left(\lambda f + \mu g\big|_{X_{r+1}=\cdots=X_n=0}\right) \end{aligned}$$

From Proposition 5.2, we know that Sgn_1 is a locally constant map at points $(\lambda, \mu) \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{(\lambda_0, \mu_0)\}$ where $\lambda f + \mu g\big|_{X_{r+1}=\cdots=X_n=0}$ is a rank r quadratic form in the variables $X_1, \ldots, X_r$. Since $\lambda_0 f + \mu_0 g$ is a nonsingular form in r variables, we can find $\varepsilon > 0$ such that

$$\operatorname{Sgn}_1(\lambda, \mu) = \operatorname{Sgn}_1(\lambda_0, \mu_0) = \operatorname{Sgn}(\lambda_0, \mu_0)$$

for all $(\lambda, \mu) \in \mathcal{B}_\varepsilon(\lambda_0, \mu_0)$ in $\mathbb{S}^1$. Choose $(\lambda, \mu) \in \mathcal{B}_\varepsilon$ different from (λ_0, μ_0). After performing row and column operations on the symmetric matrix $M_{\lambda f + \mu g}$, it can be written in the form

$$\begin{array}{cc} & \begin{array}{cc} r & n-r \end{array} \\ \begin{array}{c} r \\ \\ n-r \end{array} & \left(\begin{array}{ccc|c} c_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & c_r & \\ \hline & 0 & & B \end{array}\right) \end{array}$$

As observed from the above matrix,

$$\operatorname{Sgn}(\lambda,\mu) = \operatorname{Sgn}_1(\lambda,\mu) + \operatorname{sgn}(B) = \operatorname{Sgn}(\lambda_0,\mu_0) + \operatorname{sgn}(B).$$

Since $|\operatorname{sgn}(B)| \leq n-r$, we obtain

$$|\operatorname{Sgn}(\lambda,\mu) - \operatorname{Sgn}(\lambda_0,\mu_0)| \leq n-r.$$

Choose $(\lambda_1,\mu_1) \in \mathcal{C}_1$ and $(\lambda_2,\mu_2) \in \mathcal{C}_2$ such that (λ_1,μ_1) and (λ_2,μ_2) lie in $\mathcal{B}_\varepsilon$. Then,

$$\begin{aligned}
&|\operatorname{Sgn}(\lambda_1,\mu_1) - \operatorname{Sgn}(\lambda_2,\mu_2)| \\
&= |\operatorname{Sgn}(\lambda_1,\mu_1) - \operatorname{Sgn}(\lambda_0,\mu_0) + \operatorname{Sgn}(\lambda_0,\mu_0) - \operatorname{Sgn}(\lambda_2,\mu_2)| \\
&\leq |\operatorname{Sgn}(\lambda_1,\mu_1) - \operatorname{Sgn}(\lambda_0,\mu_0)| + |\operatorname{Sgn}(\lambda_0,\mu_0) - \operatorname{Sgn}(\lambda_2,\mu_2)| \\
&\leq (n-r) + (n-r) = 2(n-r)
\end{aligned}$$

This finishes the proof of the Proposition. □

Example 5.4. Let

$$\begin{aligned}
f &= X_1^2 + \cdots + X_m^2 + a_{m+1}X_{m+1}^2 + \cdots + a_nX_n^2 \\
g &= \qquad\qquad\qquad\quad b_{m+1}X_{m+1}^2 + \cdots + b_nX_n^2.
\end{aligned}$$

Let $\operatorname{sgn}(g) = c$. Let $\varepsilon > 0$ be a real number. Then for sufficiently small ε, we have $\operatorname{sgn}(g+\varepsilon f) = c+m$ and $\operatorname{sgn}(g-\varepsilon f) = c-m$. Then the difference of the two signatures, which is $2m$, equals two times the dimension of the radical of g.

6. Forms in the pencil containing many hyperbolic planes

Recall that K denotes an arbitrary field with characteristic not 2. Let $H(X,Y) \in K[X,Y]$ be a homogeneous form of degree $n \geq 1$. Then H factors in $K^{alg}[X,Y]$ as a product of linear factors and we can write $H(X,Y) = \prod_{i=1}^r L_i(X,Y)^{e_i}$ where each $e_i \geq 1$, $L_i = \alpha_iX + \beta_iY \in K^{alg}[X,Y]$ is a linear form, $1 \leq i \leq r$, and $L_1,\ldots,L_r$ are distinct in the sense that if $i \neq j$, then L_i and L_j are not scalar multiples of each other. This is the same as saying that L_i and L_j are linearly independent if $i \neq j$.

Suppose that $\gamma,\delta \in K^{alg}$, $(\gamma,\delta) \neq (0,0)$, and $H(\gamma,\delta) = 0$. Then $L_i(\gamma,\delta) = 0$ for some unique value of i. In general, we say that (γ,δ) is a zero of $H(X,Y)$ of multiplicity e if $L(\gamma,\delta) = 0$ for some linear form L where L^e is the exact power of L dividing H.

Lemma 6.1. *Let f, $g \in K[X_1,\ldots,X_n]$ be quadratic forms. Assume that the homogeneous polynomial $D(\lambda,\mu) = \det(\lambda f + \mu g)$ of degree n does not vanish identically on K^{alg}. If (λ_0,μ_0) is a zero of $D(\lambda,\mu)$ over K^{alg} of multiplicity m and r is the rank of the quadratic form $\lambda_0 f + \mu_0 g$, then $m \geq n-r$.*

Proof. Since the homogeneous polynomial $D(\lambda,\mu)$ does not vanish on K^{alg}, $D(\lambda,\mu)$ has only finitely many linear forms (up to scalar multiplication) that occur as factors of D. Let (λ_0,μ_0) be a nontrivial zero of $D(\lambda,\mu)$. We can assume that $\mu_0 \neq 0$. After an invertible linear change of variables, we can diagonalize and rewrite $\lambda_0 f + \mu_0 g$ as

$$\lambda_0 f + \mu_0 g = b_1X_1^2 + \cdots + b_rX_r^2,$$

where $\operatorname{rank}(\lambda_0 f + \mu_0 g) = r < n$. Then

$$\begin{aligned}\lambda f + \mu g &= \lambda f + \mu \frac{\lambda_0 f + \mu_0 g - \lambda_0 f}{\mu_0} \\ &= \left(\lambda - \frac{\lambda_0}{\mu_0}\mu\right) f + \frac{\mu}{\mu_0}(b_1 X_1^2 + \cdots + b_r X_r^2)\end{aligned}$$

Let M denote the symmetric matrix corresponding to the quadratic form $\lambda f + \mu g$. Then $D(\lambda, \mu) = \det(M)$, where the matrix M is as shown below and $\alpha = \left(\lambda - \frac{\lambda_0}{\mu_0}\mu\right)$:

$$\begin{array}{cc} & \begin{array}{cc} \quad\quad r \quad\quad & \quad\quad n-r \quad\quad \end{array} \\ \begin{array}{c} r \\ \\ n-r \end{array} & \left(\begin{array}{ccc|ccc} \alpha a_{11} + \frac{\mu b_1}{\mu_0} & & * & & & \\ & \ddots & & & * & \\ * & & \alpha a_{rr} + \frac{\mu b_r}{\mu_0} & & & \\ \hline & & & \alpha a_{r+1,r+1} & & * \\ & * & & & \ddots & \\ & & & * & & \alpha a_{nn} \end{array}\right) \end{array}$$

Each term of the last $n-r$ rows of M contains a factor of $\alpha = \left(\lambda - \frac{\lambda_0}{\mu_0}\mu\right)$. This implies that $\left(\lambda - \frac{\lambda_0}{\mu_0}\mu\right)^{n-r}$ divides $D(\lambda, \mu)$ over K^{alg}. Thus the linear factor $(\mu_0 \lambda - \lambda_0 \mu)$ appears at least $n-r$ times in the linear factor decomposition of $D(\lambda, \mu)$ over K^{alg}. Therefore, $m_{(\lambda_0, \mu_0)} \geq n - r$. □

For $x \in \mathbb{R}$, recall that $\lceil x \rceil$ denotes the least integer greater than or equal to x.

Lemma 6.2. *Let f, $g \in K[X_1, \ldots, X_n]$ be quadratic forms such that the determinant polynomial $D(\lambda, \mu) = \det(\lambda f + \mu g)$ over K is not identically zero. Let L be an extension of K with $K \subseteq L \subseteq K^{alg}$, and let $r \leq \left\lceil \frac{n}{2} \right\rceil$ be a positive integer. Then the following statements are equivalent.*

(a) *Every form in $\mathcal{P}_K(f, g)$ has rank at least r.*
(b) *Every form in $\mathcal{P}_{K^{alg}}(f, g)$ has rank at least r.*
(c) *Every form in $\mathcal{P}_L(f, g)$ has rank at least r.*

Proof. Since $K \subseteq L \subseteq K^{alg}$, if every form in $\mathcal{P}_{K^{alg}}(f, g)$ has rank at least r, then every form in $\mathcal{P}_L(f, g)$ has rank at least r, which further implies that every form in $\mathcal{P}_K(f, g)$ has rank at least r. This shows that (b) ⇒ (c) ⇒ (a). We finish the proof by showing that (a) ⇒ (b).

Suppose that every form in $\mathcal{P}_K(f, g)$ has rank at least r, and suppose that there exists a form $\alpha f + \beta g$ in $\mathcal{P}_{K^{alg}}(f, g)$ such that

$$\operatorname{rank}(\alpha f + \beta g) \leq r - 1.$$

We can assume that either $\alpha = 1$ or $\beta = 1$ because $(\alpha, \beta) \neq (0, 0)$ and (α, β) can be multiplied by any nonzero element of K^{alg}. Assume that $\alpha = 1$. (The other case is handled similarly.) Then $(1, \beta)$ is a zero of the determinant polynomial $D(\lambda, \mu)$ and Lemma 6.1 implies that

$$m_{(1,\beta)} \geq n - (r - 1) \geq n - \left(\left\lceil \frac{n}{2} \right\rceil - 1\right) = \begin{cases} \dfrac{n}{2} + 1, & \text{if } n \text{ is even} \\ \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \end{cases} > \frac{n}{2}.$$

The same inequality holds for each conjugate of $(1, \beta)$. Since the degree of $D(\lambda, \mu)$ is n, it follows that $(1, \beta)$ has only one conjugate, and thus $\beta \in K$, a contradiction to (a). Hence every form in $\mathcal{P}_{K^{alg}}(f, g)$ has rank at least r. $\square$

Example 6.3. This example shows that Lemma 6.2 fails to hold if $r > \lceil \frac{n}{2} \rceil$. Let $f = X_1^2 - X_2^2$ and $g = 2X_1X_2$. Then each form in $\mathcal{P}_{\mathbb{R}}(f, g)$ has rank $2 > \lceil \frac{n}{2} \rceil$ because $\det(\lambda f + \mu g) = \det \begin{pmatrix} \lambda & \mu \\ \mu & -\lambda \end{pmatrix} = -(\lambda^2 + \mu^2)$. But over $\mathbb{C}$, there are two forms of rank 1. Namely, $f + ig = (X_1 + iX_2)^2$ and $f - ig = (X_1 - iX_2)^2$.

Proposition 6.4. *Let $f, g \in \mathbb{R}[X_1, \ldots, X_n]$ be quadratic forms and assume that at least one of f, g has rank n. Suppose that every form in $\mathcal{P}_{\mathbb{R}}(f, g)$ has rank at least r. Then there exists a rank n form in $\mathcal{P}_{\mathbb{R}}(f, g)$ that splits off at least $\left\lceil \frac{r}{2} \right\rceil$ hyperbolic planes over $\mathbb{R}$, where*

$$\left\lceil \frac{r}{2} \right\rceil = \begin{cases} \dfrac{r}{2} & \text{if } r \text{ is even} \\ \dfrac{r+1}{2} & \text{if } r \text{ is odd.} \end{cases}$$

Proof. Assume that no rank n form $\lambda f + \mu g$ in $\mathcal{P}_{\mathbb{R}}(f, g)$ splits off $\lceil \frac{r}{2} \rceil$ hyperbolic planes. Let $h \in \mathcal{P}_{\mathbb{R}}(f, g)$ be any form having rank n and suppose that h splits off exactly j hyperbolic planes where $j \leq \lceil \frac{r}{2} \rceil - 1$. Then Lemma 2.7 implies that

$$h \cong X_1X_2 + \cdots + X_{2j-1}X_{2j} + h'(X_{2j+1}, \ldots, X_n),$$

where h' is definite. Thus

$$|\operatorname{sgn}(h)| = |\operatorname{sgn}(h')| = n - 2j \geq n - 2\left(\left\lceil \frac{r}{2} \right\rceil - 1\right)$$

for any form $h \in \mathcal{P}_{\mathbb{R}}(f, g)$ having rank n. Let

$$T = \{(\lambda, \mu) \in \mathbb{S}^1 \mid \det(\lambda f + \mu g) = 0\},$$

and let $\mathcal{C}_i$, $1 \leq i \leq t$, denote the connected components in $\mathbb{S}^1 - T$. Since Sgn is an odd function, there are two adjacent connected components on $\mathbb{S}^1$ where the signature jumps from being positive to negative or vice versa. Therefore, there must be a jump of at least $2\left(n - 2\left(\left\lceil \frac{r}{2} \right\rceil - 1\right)\right)$ for the signature as (λ, μ) varies on $\mathbb{S}^1$. By Proposition 5.3, such a jump occurs only when (λ, μ) passes through a point in T, and the jump is bounded above by twice the dimension of the radical of the associated singular form. Let $\lambda_0 f + \mu_0 g$ be that singular form in $\mathcal{P}_{\mathbb{R}}(f, g)$ and let $r_0 = \operatorname{rank}(\lambda_0 f + \mu_0 g)$. Then the jump in the signature as we pass through (λ_0, μ_0) is bounded above by $2(n - r_0)$. Therefore,

$$\begin{aligned} 2\left(n - 2\left(\left\lceil \frac{r}{2} \right\rceil - 1\right)\right) &\leq 2(n - r_0) \\ -2\left(\left\lceil \frac{r}{2} \right\rceil - 1\right) &\leq -r_0 \\ r_0 \leq 2\left(\left\lceil \frac{r}{2} \right\rceil - 1\right) &< r, \end{aligned}$$

which is a contradiction because every form in $\mathcal{P}_{\mathbb{R}}(f, g)$ has rank at least r. Hence there exists $(\lambda, \mu) \in \mathbb{S}^1$ such that $\operatorname{rank}(\lambda f + \mu g) = n$ and splits off at least $\left\lceil \frac{r}{2} \right\rceil$ hyperbolic planes. $\square$

Example 6.5. Let $n \geq 2$ and let

$$\begin{aligned} f &= r_1X_1^2 + r_2X_2^2 + \cdots + r_nX_n^2 \\ g &= X_1^2 + X_2^2 + \cdots + X_n^2. \end{aligned}$$

Assume that $r_1, \dots, r_n \in \mathbb{R}$ and $r_1 < r_2 < \cdots < r_n$. If n is even, choose $t \in \mathbb{R}$ such that $r_{\frac{n}{2}} < t < r_{\frac{n}{2}+1}$. If n is odd, choose $t \in \mathbb{R}$ such that either $r_{\frac{n-1}{2}} < t < r_{\frac{n+1}{2}}$ or $r_{\frac{n+1}{2}} < t < r_{\frac{n+1}{2}+1}$. Let $h = f - tg$. Then h has rank n and

$$\operatorname{sgn}(h) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 \text{ or } -1 & \text{if } n \text{ is odd.} \end{cases}$$

Every form in the $\mathbb{R}$-pencil of f, g has rank $\geq n-1$ and h splits off exactly $\lceil \frac{n-1}{2} \rceil$ hyperbolic planes. This shows that the bound in Proposition 6.4 is optimal.

Example 6.6. Let K be a field with $\operatorname{char} K \neq 2$ and let $n = 2m+1$, $m \geq 1$. Let

$$\begin{aligned} f_m &= X_1X_2 + X_3X_4 + \cdots + X_{2m-1}X_{2m} \\ g_m &= \quad X_2X_3 + X_4X_5 + \quad \cdots \quad + X_{2m}X_{2m+1}. \end{aligned}$$

Every quadratic form in $\mathcal{P}_K(f_m, g_m)$ can be written

$$af_m + bg_m = X_2(aX_1 + bX_3) + \cdots + X_{2m}(aX_{2m-1} + bX_{2m+1}).$$

Thus every such form in $\mathcal{P}_K(f_m, g_m)$ has rank $2m$. Every quadratic form in $\mathcal{P}_K(f_m, g_m)$ splits off exactly $\frac{2m}{2} = m$ hyperbolic planes. Note that $D(\lambda, \mu) = \det(\lambda f_m + \mu g_m) = 0$ and that no form in $\mathcal{P}_K(f_m, g_m)$ has rank n.

Pairs of quadratic forms, such as those in Example 6.6 and Theorem 6.7, are essential for classifying pairs of quadratic forms over fields K with $\operatorname{char} K \neq 2$. We state Theorem 6.7 without proof, but the interested reader can find a proof and additional details in [5, Theorem 3.3] and [9, Theorems 3.1, 3.3].

Theorem 6.7. *Let K be an infinite field with $\operatorname{char} K \neq 2$. Let $f, g \in K[X_1, \dots, X_n]$ be quadratic forms and assume that $\operatorname{rad}(f) \cap \operatorname{rad}(g) = 0$. Then there exist uniquely defined positive integers $m_1, \dots, m_j$, $j \geq 0$, such that the pair f, g is equivalent to*

$$f \cong f_{m_1} \perp \cdots \perp f_{m_j} \perp q_2(X_{M+1}, \dots, X_{M+N})$$

$$g \cong g_{m_1} \perp \cdots \perp g_{m_j} \perp q_2'(X_{M+1}, \dots, X_{M+N}),$$

and such that the determinant polynomial $D(\lambda, \mu) = \det(\lambda q_2 + \mu q_2')$ over K is not identically zero, $M = \sum_{i=1}^{j}(2m_i + 1)$, and $M + N = n$.

Theorem 6.7 allows us to prove a stronger version of Proposition 6.4 where we can weaken the hypothesis and still conclude the same result.

Theorem 6.8. *Let $f, g \in \mathbb{R}[X_1, \dots, X_n]$ be quadratic forms and assume that $\operatorname{rad}(f) \cap \operatorname{rad}(g) = 0$. Suppose that every form in $\mathcal{P}_{\mathbb{R}}(f, g)$ has rank at least r. Then there exists a form in $\mathcal{P}_{\mathbb{R}}(f, g)$ that splits off at least $\left\lceil \frac{r}{2} \right\rceil$ hyperbolic planes over $\mathbb{R}$, where*

$$\left\lceil \frac{r}{2} \right\rceil = \begin{cases} \dfrac{r}{2} & \text{if } r \text{ is even} \\ \dfrac{r+1}{2} & \text{if } r \text{ is odd.} \end{cases}$$

Proof. Let $q_1 = f_{m_1} \perp \cdots \perp f_{m_j}$ and $q_1' = g_{m_1} \perp \cdots \perp g_{m_j}$ in the notation of Theorem 6.7. Every form in $\mathcal{P}_K(q_1, q_1')$ has rank $2(m_1 + \cdots + m_j)$ and splits off $m_1 + \cdots + m_j$ hyperbolic planes. The determinant polynomial $\det(\lambda q_2 + \mu q_2') \neq 0$ and only finitely many forms in $\mathcal{P}_{\mathbb{R}}(q_2, q_2')$ have rank less than N. Every form in $\mathcal{P}_{\mathbb{R}}(q_2, q_2')$ has rank at least $R := r - 2(m_1 + \cdots + m_j)$. By Proposition 6.4, there exists a rank N form in $\mathcal{P}_{\mathbb{R}}(q_2, q_2')$ that splits off at least $\lceil \frac{R}{2} \rceil$ hyperbolic planes. Therefore, there exists a form in $\mathcal{P}_{\mathbb{R}}(f, g)$ that splits off at least $(m_1 + \cdots + m_j) + \lceil \frac{R}{2} \rceil = \lceil \frac{r}{2} \rceil$ hyperbolic planes. $\square$

Remark 6.9. If a quadratic form in n variables splits off k hyperbolic planes, then $2k \leq n$. Thus Proposition 6.4 implies that $2\left\lceil \frac{r}{2} \right\rceil \leq n$. If r is even, this gives $r \leq n$. If r is odd, this gives $r+1 \leq n$, so $r \leq n-1$. That is, if r is odd, it is not possible that every form in $\mathcal{P}_{\mathbb{R}}(f,g)$ has rank $r = n$. Here is a way to see this directly. Suppose that $r = n$ is odd. Then $D(\lambda,\mu) = \det(\lambda f + \mu g)$ is a homogeneous form of odd degree n over $\mathbb{R}$. Since every form of odd degree over $\mathbb{R}$ has at least one nontrivial zero, there exist $\lambda_0, \mu_0 \in \mathbb{R}$, not both zero, such that $\lambda_0 f + \mu_0 g$ is singular, and so $\operatorname{rank}(\lambda_0 f + \mu_0 g) < n$. Therefore, in Proposition 6.4, if r is odd, then $r < n$.

7. Nonsingular zeros and simultaneous diagonalization

Definition 7.1 (Nonsingular Zero)**.** Let $f, g \in K[X_1, \ldots, X_n]$ be quadratic forms.

(1) A vector $v \in K^n$ is a nonsingular zero of f if $f(v) = 0$, and
$$\frac{\partial f}{\partial X}(v) = \left(\frac{\partial f}{\partial X_1}(v), \ldots, \frac{\partial f}{\partial X_n}(v) \right)$$
is not the zero vector, and is a singular zero otherwise.

(2) A vector v is a nonsingular common zero of a pair of quadratic forms f, g if $f(v) = g(v) = 0$, and the vectors
$$\frac{\partial f}{\partial X}(v),\ \frac{\partial g}{\partial X}(v)$$
are linearly independent over K, and is a singular common zero otherwise.

(3) We say that f, g is a nonsingular pair of quadratic forms if every nontrivial common zero of f, g defined over K^{alg} is a nonsingular zero.

Proposition 7.2. *Let $n \geq 2$ and let $f,\ g \in K[X_1, \ldots, X_n]$ be a nonsingular pair of quadratic forms. Then every form in $\mathcal{P}_K(f,g)$ has rank at least $n-1$.*

Proof. Assume that there exists a form in $\mathcal{P}_K(f,g)$ having rank at most $n-2$. We can assume that $g = g(X_1, \ldots, X_{n-2})$. Let $h(X_{n-1}, X_n) = f(0, \ldots, 0, X_{n-1}, X_n)$. There exist $a, b \in K^{alg}$, with $(a,b) \neq (0,0)$, such that $h(a,b) = 0$. Then $(0, \ldots, 0, a, b)$ is a nontrivial singular zero of $f = g = 0$. □

For quadratic forms $f, g, \in K[X_1, \ldots, X_n]$, if $D(\lambda,\mu)$ is nonzero, then $D(\lambda,\mu)$ factors as a product of linear factors over K^{alg}. Then next result shows that the linear factors are distinct up to nonzero scalar factors in K^{alg} if and only if f, g is a nonsingular pair.

Proposition 7.3. *Let K be a field with $\operatorname{char} K \neq 2$ and let $f,\ g \in K[X_1, \ldots, X_n]$ be quadratic forms. The following statements are equivalent.*

(1) *f, g is a nonsingular pair of quadratic forms.*

(2) *The homogeneous polynomial $D(\lambda,\mu) = \det(\lambda f + \mu g)$ has no repeated linear factors over K^{alg}.*

Proof. (2) ⇒ (1). Suppose that $a = (a_1, \ldots, a_n)$ is a nontrivial singular zero of $f = g = 0$ where each $a_i \in K^{alg}$. We can assume that this singular zero has coordinates $(1, 0, \ldots, 0)$. Then
$$f = X_1 L_1(X_2, \ldots, X_n) + Q_1(X_2, \ldots, X_n),$$
where L_1 is a linear form and Q_1 is a quadratic form. Similarly,
$$g = X_1 L_2(X_2, \ldots, X_n) + Q_2(X_2, \ldots, X_n).$$
It follows from Definition 7.1 that L_1, L_2 are linearly dependent. Thus there exist $c, d \in K^{alg}$, not both zero, such that $cL_1 + dL_2 = 0$. We can assume that $d \neq 0$ by interchanging f and g if necessary. Then we can assume that $L_2 = 0$ by replacing f, g

with $f, cf+dg$. Thus λ divides each entry of the first row and column of $\lambda M_f + \mu M_g$ and its $(1,1)$-entry is zero. It follows that $\lambda^2 \mid D(\lambda,\mu)$, a contradiction.

(1) $\Rightarrow$ (2). Suppose that $D(\lambda,\mu)$ has a linear factor of multiplicity at least 2 over K^{alg}. By choosing appropriate linear combinations of f, g in place of f, g, we can assume that $\lambda^2 \mid D(\lambda,\mu)$. Then the coefficient of μ^n in $D(\lambda,\mu)$ is zero, and thus $\det(M_g)=0$, which implies that g has rank at most $n-1$. If g has rank at most $n-2$, then $f=g=0$ has a singular zero over K^{alg} by Proposition 7.2. Thus g has rank $n-1$. We can assume that $g = g(X_2,\dots,X_n)$. Then $g(1,0,\dots,0)=0$ and $(1,0,\dots,0) \in \mathrm{rad}(g)$, where $(1,0,\dots,0)$ denotes the point where $X_1 = 1$ and $X_i = 0$ for $i \geq 2$. Since the first row and column of M_g are both zero, the coefficient of $\lambda\mu^{n-1}$ in $D(\lambda,\mu)$ is given by the $(1,1)$-entry of M_f times the determinant of the lower right $(n-1)\times(n-1)$ submatrix of M_g. The coefficient of $\lambda\mu^{n-1}$ in $D(\lambda,\mu)$ is zero because λ^2 divides $D(\lambda,\mu)$. Since $\mathrm{rank}(g) = n-1$, it follows that the $(1,1)$-entry of M_f is zero and thus $f(1,0,\dots,0)=0$. Therefore, $(1,0,\dots,0)$ is a singular zero of $f=g=0$ because $(1,0,\dots,0)\in\mathrm{rad}(g)$. □

The converse of Proposition 7.2 does not hold in general, as shown in the next example, but we show in Proposition 7.8 that the converse does hold if f and g are simultaneously diagonalized as above.

Example 7.4. Let $f = 2X_1X_2$, $g = X_2^2$. Then every form in $\mathcal{P}_{\mathbb{R}}(f,g)$ has rank at least 1 but $(1,0)$ is a singular zero of the pair f, g. We have $D(\lambda,\mu) = \det(\lambda f + \mu g) = -\lambda^2$, and so $D(\lambda,\mu)$ does not have distinct linear factors, as predicted by Proposition 7.3.

Lemma 7.5. *Let $f, g \in K[X_1,\dots,X_n]$ be quadratic forms and suppose that* $\mathrm{rank}(g) <$ *n. If either f, g have no nontrivial common zero over K or $\lambda^2 \nmid D(\lambda,\mu)$, then the pair f, g is equivalent over K to*

$$\begin{aligned} f &= a_1X_1^2 + f_1(X_2,\dots,X_n) \\ g &= \phantom{a_1X_1^2 + {}} g_1(X_2,\dots,X_n), \end{aligned}$$

where $a_1 \neq 0$.

Proof. There is an invertible linear change of variables over K that lets us assume that $g = g(X_2,\dots,X_n)$. Let $P = (1,0,\dots,0)$. We can write

$$f = a_1X_1^2 + X_1L(X_2\dots,X_n) + Q(X_2,\dots,X_n)$$

where $L, Q \in K[X_2,\dots,X_n]$ with L a linear form and Q a quadratic form.

Suppose that $a_1 = 0$. Then $f(P) = a_1 = 0$ and $g(P) = 0$. In addition, a straightforward computation shows that $\lambda^2 \mid D(\lambda,\mu)$. One of these statements contradicts our hypotheses, and thus $a_1 \neq 0$.

We have

$$f = a_1\left(X_1 + \frac{1}{2a_1}L\right)^2 + Q - \frac{1}{4a_1}L^2.$$

Let $X_1' = X_1 + \frac{1}{2a_1}L$ and $f_1 = Q - \frac{1}{4a_1}L^2$. Then $f = a_1(X_1')^2 + f_1$ and so the pair f, g is equivalent to

$$\begin{aligned} f &= a_1X_1^2 + f_1(X_2,\dots,X_n) \\ g &= \phantom{a_1X_1^2 + {}} g_1(X_2,\dots,X_n), \end{aligned}$$

where $f_1, g_1 \in K[X_2,\dots,X_n]$ are quadratic forms. □

Theorem 7.6. *Let $f, g \in K[X_1,\dots,X_n]$ be quadratic forms and assume that $D(\lambda,\mu)$ is a product of linear factors defined over K. If either f, g have no nontrivial common zero over K or $D(\lambda,\mu)$ has no repeated linear factors, then f, g can be simultaneously diagonalized over K.*

Proof. Since $D(\lambda,\mu)=\det(\lambda f+\mu g)$ has a linear factor defined over K, some form in $\mathcal{P}_K(f,g)$ has rank at most $n-1$. By choosing appropriate generators of $\mathcal{P}_K(f,g)$, we can assume that $\operatorname{rank}(g)<n$. Since either f,g have no nontrivial common zero over K or $\lambda^2\nmid D(\lambda,\mu)$, Lemma 7.5 implies that the pair f,g is equivalent over K to

$$\begin{aligned} f &= a_1X_1^2 + f_1(X_2,\dots,X_n)\\ g &= \phantom{a_1X_1^2 + {}} g_1(X_2,\dots,X_n), \end{aligned}$$

where $a_1\neq 0$.

Suppose that m is maximal such that f, g are equivalent over K to

$$\begin{aligned} f &= a_1X_1^2+\cdots+a_mX_m^2+q(X_{m+1},\dots,X_n)\\ g &= b_1X_1^2+\cdots+b_mX_m^2+q'(X_{m+1},\dots,X_n), \end{aligned}$$

where $q,q'\in K[X_{m+1},\dots,X_n]$ are quadratic forms. Then $m\geq 1$. Suppose that $m<n$.

By unique factorization in $K[\lambda,\mu]$ it follows that $\det(\lambda q+\mu q')$ is a product of linear factors defined over K. In addition, either q,q' have no nontrivial common zero defined over K or $\det(\lambda q+\mu q')$ has no repeated linear factors. Repeating the argument at the beginning of this proof gives a contradiction to the maximality of m. Therefore, $m=n$, as desired. ☐

The next result shows that a nonsingular pair of quadratic forms can always be simultaneously diagonalized over an algebraically closed field.

Proposition 7.7. *Let $f,g\in K[X_1,\dots,X_n]$ be a nonsingular pair of quadratic forms. Then f,g can be simultaneously diagonalized over K^{alg}.*

Proof. Proposition 7.3 implies that $D(\lambda,\mu)=\det(\lambda f+\mu g)$ is a product of distinct linear factors over K^{alg} up to nonzero scalar factors in K. The previous theorem implies that f,g can be simultaneously diagonalized over K^{alg}. ☐

Suppose that $f,g\in K[X_1,\dots,X_n]$ are simultaneously diagonalized quadratic forms. Then

$$\begin{aligned} f &= a_1X_1^2+a_2X_2^2+\cdots+a_nX_n^2\\ g &= b_1X_1^2+b_2X_2^2+\cdots+b_nX_n^2, \end{aligned}$$

where each $a_i,b_j\in K$. Assume that $(a_i,b_i)\neq(0,0)$ for each i. Then

$$D(\lambda,\mu)=\det(\lambda f+\mu g)=\prod_{i=1}^{n}(\lambda a_i+\mu b_i)$$

is a nonzero homogeneous form of degree n.

The next result gives additional characterizations of a nonsingular pair of quadratic forms f,g in the case that f,g are simultaneously diagonalized.

Proposition 7.8. *Suppose that $f,g\in K[X_1,\dots,X_n]$ are simultaneously diagonalized quadratic forms, as above, with $(a_i,b_i)\neq(0,0)$ for each i. Then the following statements are equivalent.*

(1) *f,g is a nonsingular pair of quadratic forms.*
(2) *$D(\lambda,\mu)$ has no repeated linear factors over K^{alg}.*
(3) *Every form in $\mathcal{P}_K(f,g)$ has rank at least $n-1$.*
(4) *$a_ib_j-a_jb_i\neq 0$ for every $i\neq j$.*

Proof. We have already seen in Proposition 7.3 that (1) and (2) are equivalent, and in Proposition 7.2 that (1) $\Rightarrow$ (3). We will prove that the negations of (2), (3), (4) are equivalent. Statement (2) is false $\Leftrightarrow$ there exists $c\in K^\times$ and $i\neq j$ such that $c(\lambda a_i+\mu b_i)=\lambda a_j+\mu b_j$ $\Leftrightarrow$ there exists $c\in K^\times$ and $i\neq j$ such that $a_j=ca_i$ and $b_j=cb_i$

$\Leftrightarrow$ there exists $c \in K^\times$ and $i \neq j$ such that $\begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix} \begin{pmatrix} c \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow$ statement (4) is false. The last equivalence uses the assumption that each $(a_i, b_i) \neq (0,0)$.

Statement (3) is false $\Leftrightarrow$ there exist $r, s \in K$, not both zero, such that $rf + sg$ has rank at most $n-2$ $\Leftrightarrow$ there exist $r, s \in K$, not both zero, and $i \neq j$ such that $ra_i + sb_i = ra_j + sb_j = 0$ $\Leftrightarrow$ there exist $r, s \in K$, not both zero, and $i \neq j$ such that $\begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow$ statement (4) is false. $\square$

Let $A = \begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}$. Note that the proof of the equivalence of (2) and (4) used the matrix A, and the proof of the equivalence of (3) and (4) used the matrix A^t.

If $\operatorname{char} K = 2$, then the characterization of nonsingular pairs of quadratic forms is much more difficult. See [6].

8. Simultaneous diagonalization over the real numbers and the Spectral Theorem

In this section, we include additional results on nonsingular pairs of quadratic forms and simultaneous diagonalization that are specific to $\mathbb{R}$.

Proposition 8.1. *Let $f, g \in \mathbb{R}[X_1, \ldots, X_n]$ be a nonsingular pair of quadratic forms. Then there exists a nonsingular form in $\mathcal{P}_{\mathbb{R}}(f,g)$ that splits off $\lceil \frac{n-1}{2} \rceil$ hyperbolic planes over $\mathbb{R}$.*

Proof. Since every nontrivial zero of $f = g = 0$ is nonsingular, Proposition 7.2 implies that each form in $\mathcal{P}_{\mathbb{R}}(f,g)$ has rank at least $n-1$. Proposition 6.4 implies that there exists a nonsingular form in $\mathcal{P}_{\mathbb{R}}(f,g)$ that splits off at least $\lceil \frac{n-1}{2} \rceil$ hyperbolic planes over $\mathbb{R}$. Since $2(\lceil \frac{n-1}{2} \rceil + 1) > n$, it follows that this nonsingular form splits off exactly $\lceil \frac{n-1}{2} \rceil$ hyperbolic planes over $\mathbb{R}$. $\square$

Remark 8.2. If $n = 8$, Proposition 8.1 implies that there is a nonsingular form in $\mathcal{P}_{\mathbb{R}}(f,g)$ that splits off 4 hyperbolic planes. This strengthens a result in [3, Lemma 12.1], where it is proved that there exists a nonzero form in $\mathcal{P}_{\mathbb{R}}(f,g)$ that splits off at least 3 hyperbolic planes.

Lemma 8.3. *Let $f, g \in \mathbb{R}[X_1, \ldots, X_n]$ be quadratic forms. Suppose that $D(\lambda, \mu) = \det(\lambda f + \mu g)$ has no linear factors defined over $\mathbb{R}$. Then every form $h \in \mathcal{P}_{\mathbb{R}}(f,g)$ has rank n, n is even, and $\operatorname{sgn}(h) = 0$.*

Proof. The hypothesis implies that $n \geq 2$. Proposition 5.2 implies that $\operatorname{sgn}(h)$ is the same for all $h \in \mathcal{P}_{\mathbb{R}}(f,g)$ because T (as defined prior to Lemma 5.1) is the empty set. Since $\operatorname{sgn}(-h) = -\operatorname{sgn}(h)$, it follows that each $h \in \mathcal{P}_{\mathbb{R}}(f,g)$ has $\operatorname{sgn}(h) = 0$. The hypothesis implies that every form in $\mathcal{P}_{\mathbb{R}}(f,g)$ has rank n. Thus n is even. $\square$

Theorem 8.4. *If $f, g \in \mathbb{R}[X_1, \ldots, X_n]$ are quadratic forms and f is definite, then $D(\lambda, \mu)$ is a product of linear factors defined over $\mathbb{R}$ and f, g can be simultaneously diagonalized over $\mathbb{R}$.*

Proof. The proof is by induction on $n \geq 1$, with the case $n = 1$ being obvious. Assume that $n \geq 2$ and that the result is true for fewer than n variables.

First suppose that $D(\lambda, \mu)$ has no linear factor defined over $\mathbb{R}$. Then Lemma 8.3 implies that each $h \in \mathcal{P}_{\mathbb{R}}(f,g)$ has rank n and signature 0. This is a contradiction because f is definite and thus has signature $\pm n$. Then some form in $\mathcal{P}_{\mathbb{R}}(f,g)$ has rank at most $n-1$. By choosing appropriate generators of $\mathcal{P}_{\mathbb{R}}(f,g)$, we can assume that $\operatorname{rank}(g) < n$. Lemma 7.5 implies that f, g is equivalent to the pair

$$\begin{aligned} f &= a_1X_1^2 + f_1(X_2, \ldots, X_n) \\ g &= \qquad\quad\; g_1(X_2, \ldots, X_n), \end{aligned}$$

where $a_1 \neq 0$. The pair f_1, g_1 satisfies the hypotheses because f_1 is definite. By induction it follows that $\det(\lambda f_1 + \mu g_1)$ is a product of linear factors defined over $\mathbb{R}$ and f_1, g_1 can be simultaneously diagonalized over $\mathbb{R}$. Then the same holds for f, g. □

Corollary 8.5. *If $n \geq 3$, and $f, g \in \mathbb{R}[X_1, \ldots, X_n]$ are quadratic forms having no nontrivial common zero over $\mathbb{R}$, then f, g can be simultaneously diagonalized over $\mathbb{R}$.*

Proof. Since $n \geq 3$ and f, g have no nontrivial common zero defined over $\mathbb{R}$, Proposition 4.4 implies that there is a form $h \in \mathcal{P}_{\mathbb{R}}(f,g)$ that is definite. The result now follows from the previous theorem. □

Example 8.6. This example shows that the previous corollary is false when $n = 2$. Let $f = 2X_1X_2$ and $g = X_1^2 - X_2^2$. The pair f, g is an anisotropic pair because there is no nontrivial common zero of f, g with either $X_1 = 0$ or $X_2 = 0$. The pair cannot be simultaneously diagonalized because $D(\lambda, \mu) = \det(\lambda f + \mu g) = -(\lambda^2 + \mu^2)$ has no linear factors defined over $\mathbb{R}$.

These results lead to a proof of the Spectral Theorem. Much of the proof below is standard linear algebra but is included for the sake of completeness.

Theorem 8.7 (Spectral Theorem)**.** *Let A be an $n \times n$ symmetric matrix with entries in $\mathbb{R}$. Then the following statements hold.*

(1) *Every eigenvalue of A is real.*
(2) *There exists an $n \times n$ orthogonal matrix P with entries in $\mathbb{R}$ such that P^tAP is a diagonal matrix.*
(3) *A has n pairwise orthogonal (and thus linearly independent) eigenvectors in $\mathbb{R}^n$.*

Proof. Let $g(X_1, \ldots, X_n) = X^tAX$ where X is the column vector $(X_1, \ldots, X_n)^t$ and let $f(X_1, \ldots, X_n) = X^tI_nX$ where I_n is the $n \times n$ identity matrix. Thus $f = X_1^2 + \cdots + X_n^2$. Since f is positive definite, the pair f, g can be simultaneously diagonalized over $\mathbb{R}$ by Theorem 8.4. Thus there exists an $n \times n$ invertible matrix M with entries in $\mathbb{R}$ such that $M^tAM = D_1$ and $M^tI_nM = D_2$ are both diagonal matrices. Since D_2 must be positive definite, each entry on the main diagonal of D_2 is positive. Then $D_2 = D_3^2$ for some invertible diagonal matrix D_3 with entries in $\mathbb{R}$. Let $P = MD_3^{-1}$. Then $P^tAP = (D_3^{-1})^tM^tAMD_3^{-1} = D_3^{-1}D_1D_3^{-1} = D_2^{-1}D_1$ is a diagonal matrix, and similarly, $P^tI_nP = D_3^{-1}D_2D_3^{-1} = I_n$. Thus $P^tP = I_n$ and so P is an orthogonal matrix. It follows that the columns of P are pairwise orthogonal and thus linearly independent.

We have $P^{-1}AP = P^tAP = D_2^{-1}D_1$, which gives $AP = P(D_2^{-1}D_1)$. Let $D_2^{-1}D_1 = \operatorname{diag}(d_1, \ldots, d_n)$. Let $v_1, \ldots, v_n$ be the columns of P. The equation $AP = P(D_2^{-1}D_1)$ implies that $Av_i = d_iv_i$ for $1 \leq i \leq n$. Thus the columns of P are pairwise orthogonal eigenvectors of A with eigenvalues $d_1, \ldots, d_n$. These eigenvalues of A are real because the diagonal matrices D_1 and D_2 have real entries. □

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