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PROBLEMS

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Problems

The first problem this issue was posed by Anthony O'Farrell, editor of this Bulletin.

Problem 88.1. Consider the sequence $x_0, x_1, \dots$ defined by $x_0 = \sqrt{5}$ and $x_n = \sqrt{2 + x_{n-1}}$, for $n = 1, 2, \dots$. Prove that

$$\prod_{n=1}^{\infty} \frac{2}{x_n} = 2\log\left(\frac{1+\sqrt{5}}{2}\right).$$

The second problem is courtesy of J.P. McCarthy of Munster Technological University.

Problem 88.2. Let P be a 3-by-3 matrix each entry of which is an n-by-n complex Hermitian matrix; that is, each entry P_{ij} is an n-by-n complex matrix equal to its own conjugate transpose P_{ij}^*. Suppose that the sum along any row or column of P is the n-by-n identity matrix I_n:

$$\sum_{k=1}^{3} P_{kj} = \sum_{k=1}^{3} P_{ik} = I_n.$$

Suppose also that the entries of P along rows and columns satisfy

$$P_{ik}P_{il} = \delta_{kl}P_{ik} \quad \text{and} \quad P_{kj}P_{lj} = \delta_{kl}P_{kj},$$

where δ_{kl} is 1 if k and l are equal and otherwise it is 0 (and no summation convention should be applied). Prove that the entries of P commute with one another.

The third problem comes from Seán Stewart of the King Abdullah University of Science and Technology, Saudi Arabia.

Problem 88.3. Prove that

$$\sum_{n=2}^{\infty} \frac{(-1)^n H_{\lfloor n/2 \rfloor}}{n} = (\log 2)^2,$$

where H_n denotes the nth harmonic number

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

and $\lfloor\cdot\rfloor$ denotes the floor function.

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Solutions

Here are solutions to the problems from *Bulletin* Number 86.

The first problem was solved by Eugene Gath of the University of Limerick, Daniel Văcaru of Pitești, Romania, the North Kildare Mathematics Problem Club, and the proposer, Yagub Aliyev of ADA University, Azerbaijan. We present the solution of Eugene Gath.

Problem 86.1. Find the nearest integer to

$$10^{2021} - \sqrt{(10^{2021})^2 - 10^{2021}}.$$

Solution 86.1. Let $f(n) = n - \sqrt{n^2 - n}$, for each positive integer n. Observe that

$$n - \sqrt{n^2 - n} = \frac{n}{n + \sqrt{n^2 - n}}.$$

Hence

$$f(n) > \frac{n}{n+n} = \frac{1}{2} \quad \text{and} \quad f(n) \leqslant \frac{n}{n} = 1.$$

Consequently, the nearest integer to $f(n)$ is 1 for all values of n including $n = 10^{2021}$. $\square$

Yagub observes that the continuous function $g(x) = 10^x - \sqrt{10^{2x} - 10^x}$ $(x > 0)$ behind this question tests the limits of graphing software, with most software unable to plot the graph of g accurately beyond about $x = 15$.

The next problem was solved by Prithwijit De of HBCSE, Mumbai, India, Eugene Gath, the North Kildare Mathematics Problem Club and the proposer, Seán Stewart. The solution we present is that of Prithwijit De.

Problem 86.2. Evaluate

$$\int_0^1 \frac{1}{x} \arctan\left(\frac{2rx}{1+x^2}\right) dx,$$

where r is a real constant.

Solution 86.2. Define $f \colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(r) = \int_0^1 \frac{1}{x} \arctan\left(\frac{2rx}{1+x^2}\right) dx.$$

Then $f(0) = 0$ and f is differentiable on $\mathbb{R}$ with derivative

$$f'(r) = \int_0^1 \frac{2(1+x^2)}{(1+x^2)^2 + (2rx)^2}\, dx.$$

Substituting $x = \tan(\theta/2)$ gives

$$f'(r) = \int_0^{\pi/2} \frac{1}{1 + r^2 \sin^2\theta}\, d\theta = \frac{\pi}{2\sqrt{1+r^2}}.$$

Integrating with respect to r gives

$$f(r) = \frac{\pi}{2} \ln\left(r + \sqrt{1+r^2}\right),$$

where we have used $f(0) = 0$ to find the constant of integration. $\square$

The third problem was solved by Henry Ricardo of the Westchester Area Math Circle, NY, USA, Ángel Plaza of the Universidad de Las Palmas de Gran Canaria, Spain, Seán Stewart, Eugene Gath, Daniel Văcaru, the North Kildare Mathematics Problem Club and the proposer, Finbarr Holland of University College Cork. Choosing between the variety of cunning solutions submitted is tricky! As usual we present only one solution, in this case the concise solution of Eugene Gath.

*Problem 85.3***.** Prove that
$$\sum_{n=0}^{\infty}\frac{9n+5}{9n^3+18n^2+11n+2}=3\log 3.$$

Solution 86.3. Observe that
$$\frac{9n+5}{9n^3+18n^2+11n+2}=3\left(\frac{1}{3n+1}+\frac{1}{3n+2}+\frac{1}{3n+3}-\frac{1}{n+1}\right).$$
Let $f(x)=3\log(1+x+x^2)$, for $x>0$. Observe that
$$f(x)=3(\log(1-x^3)-\log(1-x)),$$
for $0<x<1$. By substituting suitably into the Maclaurin series for $\log(1+y)$ we obtain
$$f(x)=3\sum_{n=0}^{\infty}\left(\frac{x^{3n+1}}{3n+1}+\frac{x^{3n+2}}{3n+2}+\frac{x^{3n+3}}{3n+3}-\frac{x^{3n+3}}{n+1}\right),$$
for $0<x<1$. This series converges at $x=1$. It follows from Abel's theorem that the series expansion for f is valid when $x=1$. Consequently,
$$3\sum_{n=0}^{\infty}\left(\frac{1}{3n+1}+\frac{1}{3n+2}+\frac{1}{3n+3}-\frac{1}{n+1}\right)=f(1)=3\log 3.$$
The result now follows from the partial fraction expansion obtained at the start. □

Finally this issue we thank Tom Barry, Chairman of New Ireland Assurance, for pointing out an error in Problem 85.2 and its solution. That problem required us to show that $2^{13}=8192$ is the only integer between 4129 and 9985 that cannot be expressed as the sum of two or more consecutive integers. However, as Tom points out, every positive integer can be expressed as the sum of two or more consecutive integers; for example,
$$8192=(-8191)+(-8190)+\cdots+8192.$$
Really the problem should have asked for the sum of two or more consecutive *positive* integers. With that addition, the solution published last issue needs some adjustments.

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer LaTeX). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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