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Injectivity

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To the memory of Edward G. Effros (1935–2019)

Abstract. The concept of dimension is ubiquitous in Mathematics. In this survey we discuss the interrelations between dimension and injectivity in the categorical sense.

1. Introduction

An invariant for an object enables us to distinguish it from a like one up to a suitable notion of isomorphism; 'dimension' is one of the most common invariants. Maybe the best known dimension is the cardinality of a basis of a vector space; which is even a complete invariant in the sense that two vector spaces (over the same field) are isomorphic if and only if they have the same dimension. In the argument that the cardinality of any two bases of a given vector space agree with each other (so that its 'dimension' is well defined) the fact that every basis of a subspace can be extended to a basis of a larger space plays an important role. This is tantamount to the statement that a linear mapping from a subspace of a vector space can always be extended to a linear mapping on the larger space; diagrammatically

$$\begin{tikzcd} E \arrow[r, "\mu"] \arrow[d, "f"'] & F \arrow[dl, dashed, "\widetilde{f}"] \\ G & \end{tikzcd} \tag{1.1}$$

where μ is the 'embedding' (an injective linear mapping) of E into F, f is the given linear mapping into a vector space G and $\widetilde{f}$ denotes the extension of f to F. In the context of modules over a (commutative, unital) ring R this is quickly seen to fail in general: given the canonical embedding $\mu\colon 2\mathbb{Z} \to \mathbb{Z}$, the $\mathbb{Z}$-linear map $f\colon 2n \mapsto n$ from $2\mathbb{Z}$ into $\mathbb{Z}$ cannot be extended to the larger module $\mathbb{Z}$ as, otherwise, the extension $\widetilde{f}$ would have to satisfy $1 = f(2) = \widetilde{f}(2) = 2\widetilde{f}(1)$ which is impossible as $\widetilde{f}(1) \in \mathbb{Z}$. The expert already notices at this stage the reason for this is that the $\mathbb{Z}$-module $\mathbb{Z}$ is not 'injective'; equivalently, the ring $\mathbb{Z}$ is not 'semisimple'.

The property of an object I in a category to ensure the 'extension' of a morphism from a 'subobject' E of an object F to F is generally called 'injectivity'; and is used in module categories to define a cohomological dimension which is not tied to the existence of a basis. In this survey article, we will review the interaction between injectivity and dimension in a wider setting of not necessarily abelian categories with a view on categories arising in functional analysis.

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The fundamental ingredients in a category are the morphisms, the 'arrows' between objects. They determine the concept of 'isomorphism' in the category, and similarly important is the choice of 'embeddings' some of which we have observed above. Typical categories we are interested in are the following ones.

Category	*Objects*	*Morphisms/Arrows*
$\mathscr{V}ec_{\mathbb{C}}$	complex vector spaces	linear maps
$\mathscr{M}od_R$	modules over a ring R	R-module maps
$\mathscr{N}or^{\infty}$	(complex) normed spaces	bounded linear maps
$\mathscr{N}or^{1}$	normed spaces	contractive linear maps
$\mathscr{B}an^{\infty}$	Banach spaces	bounded linear maps
$\mathscr{B}an^{1}$	Banach spaces	contractive linear maps
$\mathscr{T}op$	topological spaces	continuous maps

Note, however, that not all objects in a category have to be sets with some additional structure and even if they are, the morphisms need not be mappings. For instance, we could consider homotopy classes of continuous mappings between topological spaces as the morphisms. In functor categories, such as categories of sheaves, e.g., the morphisms are typically given by natural transformations.

In a category $\mathscr{A}$ we will denote by $\mathrm{obj}(\mathscr{A})$ the class of objects of $\mathscr{A}$ and, for any two $E, F \in \mathrm{obj}(\mathscr{A})$, by $\mathrm{Mor}(E,F)$ the set of all morphisms between E and F. In case we need to specify the category explicitly, we write $\mathrm{Mor}_{\mathscr{A}}(E,F)$. Let us recall some basic terminology; for a comprehensive discussion see, e.g., [1].

A morphism $f \in \mathrm{Mor}(E,F)$ is called a *monomorphism* if for any two morphisms $g, h \in \mathrm{Mor}(F,G)$ the identity $fg = fh$ implies that $g = h$, and it is called an *epimorphism* if for any two morphisms $g, h \in \mathrm{Mor}(G,E)$ the identity $gf = hf$ implies that $g = h$. In a concrete category, that is, the objects have an underlying set structure and the morphisms are set mappings (with some additional properties), every injective morphism is a monomorphism and every surjective morphism is an epimorphism however the reverse implications fail in general. The morphism $f \in \mathrm{Mor}(E,F)$ is called an *isomorphism* if there is a morphism $\bar f \in \mathrm{Mor}(F,E)$ such that $\bar f f = \mathrm{id}_E$ and $f\bar f = \mathrm{id}_F$, where id stands for the identity morphism of an object. An isomorphism is always a monomorphism and an epimorphism but the converse often fails; for example, in $\mathscr{N}or^{\infty}$.

The concept of a 'subobject' can be replaced by specifying a class $\mathscr{M}$ of monomorphisms which one usually assumes to be closed under composition and contains all isomorphisms in the category. For $E, F \in \mathrm{obj}(\mathscr{A})$ we will write

$$\mathscr{M}(E,F) = \{\mu \in \mathrm{Mor}_{\mathscr{A}}(E,F) \mid \mu \in \mathscr{M}\}$$

for the set of all morphisms between E and F that belong to the class $\mathscr{M}$. With these preparations we now introduce the main idea.

2. Injective Objects

An object $I \in \mathrm{obj}(\mathscr{A})$ is called *$\mathscr{M}$-injective* (for a specified class $\mathscr{M}$ of monomorphisms in the category $\mathscr{A}$) if, whenever $E, F \in \mathrm{obj}(\mathscr{A})$ and $\mu \in \mathscr{M}(E,F)$ are given, every $f \in \mathrm{Mor}(E,I)$ can be 'extended' to a morphism $\widetilde{f} \in \mathrm{Mor}(F,I)$, that is, $f = \widetilde{f}\mu$ as shown in the diagram (1.1) above (with $I = G$). Equivalently, if the mapping $\mu^*\colon \mathrm{Mor}(F,I) \to \mathrm{Mor}(E,I)$, $g \mapsto g\mu$ is surjective. (We shall take up this point of view in more detail in Section 3 below.)

Example 2.1. In the category $\mathscr{Nor}^1$ we choose as $\mathscr{M}$ all linear isometries. An $\mathscr{M}$-injective object I therefore has the property that, whenever E is (linearly isometric to) a subspace of a normed space F, every linear contraction from E into I can be extended to a contraction from F to I. Let $f\colon E \to I$ be a bounded linear mapping with $\|f\| \neq 0$. The contraction $f_1 = \frac{f}{\|f\|}$ is extended to a contraction $\widetilde{f_1}\colon F \to I$. With $\mu\colon E \to F$ the embedding we have $\widetilde{f_1}\mu = f_1$, equivalently, $\|f\|\widetilde{f_1}\mu = f$. Hence $\big\|\|f\|\widetilde{f_1}\big\| = \|f\|\,\|\widetilde{f_1}\| \le \|f\|$, in other words, $\widetilde{f} := \|f\|\widetilde{f_1}$ is a 'Hahn–Banach extension' of f: it has the same norm as f. The Hahn–Banach theorem now states that $\mathbb{C}$ is an $\mathscr{M}$-injective object in $\mathscr{Nor}^1$.

Taking the same class $\mathscr{M}$ in the full subcategory $\mathscr{Ban}^1$, the $\mathscr{M}$-injectives in $\mathscr{Ban}^1$ are the completions of the $\mathscr{M}$-injectives in $\mathscr{Nor}^1$.

Sometimes it is possible to characterise all injective objects in a category. For example, in $\mathscr{Ban}^1$, an object E is $\mathscr{M}$-injective (where $\mathscr{M}$ is the class of all linear isometries) if and only if E is isomorphic in $\mathscr{Ban}^1$ to a space $C(X)$ of continuous complex-valued functions on an extremally disconnected compact Hausdorff space X [11, Chapter 3, Section 11, Theorem 6].

We say that the category $\mathscr{A}$ has *enough $\mathscr{M}$-injectives* if, for every $E \in \mathrm{obj}(\mathscr{A})$, there are an $\mathscr{M}$-injective object I and a morphism $\mu \in \mathscr{M}(E, I)$; in other words, every object can be embedded into an $\mathscr{M}$-injective object.

Example 2.2. The category $\mathscr{Ban}^1$ has enough $\mathscr{M}$-injectives. The reason for this is two-fold. Firstly, every Banach space can be isometrically embedded into a space of the form

$$\ell^\infty(\Omega) = \{\varphi\colon \Omega \to \mathbb{C} \mid \varphi \text{ is bounded}\}.$$

This is a consequence of the Hahn–Banach theorem. Let $E \in \mathrm{obj}(\mathscr{Ban}^1)$ and let E_1' denote its dual unit ball, that is, the set of all bounded linear functionals on E with norm at most one. Then $x \mapsto \hat{x}$, $E \to \ell^\infty(E_1')$, where $\hat{x}(f) = f(x)$ for all $f \in E_1'$ is a linear isometry. So we may take $\Omega = E_1'$ and $\mu \in \mathscr{M}(E, \ell^\infty(\Omega))$ this isometry.

Secondly, $\mathscr{Ban}^1$ has arbitrary products, namely, for any family $\{E_\omega \mid \omega \in \Omega\}$ of Banach spaces, the space

$$\prod_{\omega\in\Omega} E_\omega = \big\{\varphi \in \mathop{\times}_{\omega\in\Omega} E_\omega \mid \sup_{\omega\in\Omega} \|\varphi(\omega)\| < \infty\big\},$$

where $\times_{\omega\in\Omega} E_\omega$ denotes the cartesian product of the family $\{E_\omega \mid \omega \in \Omega\}$. Setting $E_\omega = \mathbb{C}$ for each ω, we clearly have $\prod_{\omega\in\Omega} E_\omega = \ell^\infty(\Omega)$. Each E_ω is $\mathscr{M}$-injective (Example 2.1) and it is a general fact that products of injectives are injective in a category with products; thus $\ell^\infty(\Omega)$ is $\mathscr{M}$-injective.

Since every normed space can be isometrically embedded into a Banach space (its completion), $\mathscr{Nor}^1$ has enough $\mathscr{M}$-injectives as well.

The following terminology is useful in understanding the relations between injective and non-injective objects.

Definition 2.3. (i) Let $E, F \in \mathrm{obj}(\mathscr{A})$. We say E is a *retract of* F if there exist morphisms $s \in \mathrm{Mor}(E, F)$ and $r \in \mathrm{Mor}(F, E)$ such that $rs = \mathrm{id}_E$. In this case we call s a *section* and r a *retraction.*

(ii) An object $E \in \mathrm{obj}(\mathscr{A})$ is an *absolute $\mathscr{M}$-retract* if every $\mu \in \mathscr{M}(E, F)$ for any $F \in \mathrm{obj}(\mathscr{A})$ is a section.

Proposition 2.4. *Every $\mathscr{M}$-injective object is an absolute $\mathscr{M}$-retract. Every retract of an $\mathscr{M}$-injective object is $\mathscr{M}$-injective. If $\mathscr{A}$ has enough $\mathscr{M}$-injectives then every absolute $\mathscr{M}$-retract is $\mathscr{M}$-injective.*

Proof. Let $I \in \mathrm{obj}(\mathscr{A})$ be $\mathscr{M}$-injective and let $\mu \in \mathscr{M}(I,F)$ for some $F \in \mathrm{obj}(\mathscr{A})$. Then, for id_I, there is $r \in \mathrm{Mor}(F,I)$ such that $\mathrm{id}_I = r\mu$, so μ is a section. If $E \in \mathrm{obj}(\mathscr{A})$ and $s \in \mathrm{Mor}(E,I)$, $r \in \mathrm{Mor}(I,E)$ satisfy $rs = \mathrm{id}_E$ then, for every $f \in \mathrm{Mor}(G,E)$, G any object in $\mathscr{A}$, and $\mu \in \mathscr{M}(G,H)$, $H \in \mathrm{obj}(\mathscr{A})$, there is $\widetilde{f} \in \mathrm{Mor}(H,I)$ with $sf = \widetilde{f}\mu$ and hence, $f = \mathrm{id}_E f = rsf = r\widetilde{f}\mu$ so that E is $\mathscr{M}$-injective as a retract of I.

Suppose $\mathscr{A}$ has enough $\mathscr{M}$-injectives. Then every absolute $\mathscr{M}$-retract is a retract of an $\mathscr{M}$-injective and hence is $\mathscr{M}$-injective. □

The above result is effective in deciding which objects can be injective.

Example 2.5. Let F be a Banach space. Suppose E is a retract of F and $s\colon E \to F$ and $r\colon F \to E$ are the section and retraction, respectively. Then $(sr)^2 = srsr = sr$ is a projection of norm one from F onto E. In other words, E is a (topological) direct summand of F. As $\mathscr{B}\!an^1$ has enough $\mathscr{M}$-injectives (Example 2.2), a Banach space E is injective if and only if, whenever E is (isometrically isomorphic to) a subspace of an injective Banach space F, there is a norm-one projection from F onto E, by Proposition 2.4 above.

Let $c_0(\Omega)$ be the closed subspace of $\ell^\infty(\Omega)$ consisting of those bounded functions φ such that, for every $\varepsilon > 0$, the set $\{\omega \in \Omega \mid |\varphi(\omega)| \geq \varepsilon\}$ is finite. By a well-known result of Phillips, see, e.g., [9, Theorem 5.6], there is no bounded projection from $\ell^\infty(\Omega)$ onto $c_0(\Omega)$; as a result, $c_0(\Omega)$ is not $\mathscr{M}$-injective.

3. Additive Categories

So far the categories we considered had very few additional properties; in order to be able to define a dimension efficiently we need some more structure.

Definition 3.1. A category $\mathscr{A}$ is called *additive* if it has a zero object (a unique object 0 such that, for every $E \in \mathrm{obj}(\mathscr{A})$, both $\mathrm{Mor}_{\mathscr{A}}(E,0)$ and $\mathrm{Mor}_{\mathscr{A}}(0,E)$ are singleton sets each); for all $E,F \in \mathrm{obj}(\mathscr{A})$ the morphism set $\mathrm{Mor}_{\mathscr{A}}(E,F)$ has the structure of an (additive) abelian group (in which case it is usually denoted by $\mathrm{Hom}_{\mathscr{A}}(E,F)$) such that composition of morphisms is bilinear; and for every pair of objects $E,F \in \mathrm{obj}(\mathscr{A})$ their biproduct exists (that is, there exists $D \in \mathrm{obj}(\mathscr{A})$ together with morphisms $\mu_E \in \mathrm{Mor}_{\mathscr{A}}(E,D)$, $\pi_E \in \mathrm{Mor}_{\mathscr{A}}(D,E)$, $\mu_F \in \mathrm{Mor}_{\mathscr{A}}(F,D)$, $\pi_F \in \mathrm{Mor}_{\mathscr{A}}(D,F)$ such that $\pi_E\mu_E = \mathrm{id}_E$, $\pi_F\mu_F = \mathrm{id}_F$ and $\mu_E\pi_E + \mu_F\pi_F = \mathrm{id}_D$. In this case, the unique biproduct is usually denoted by $D = E \oplus F$ and called *the direct sum of* E and F.

More details on additive categories can be found, for example, in [12].

Example 3.2. Probably the most commonly known additive categories are module categories. Let R be a unital ring. Let $\mathscr{M}\!od_R$ denote the category whose objects are the right R-modules and the morphisms are the R-module maps (also called R-linear maps). Usually, $\mathrm{Mor}_{\mathscr{M}\!od_R}(E,F)$ is denoted by $\mathrm{Hom}_R(E,F)$, for $E,F \in \mathrm{obj}(\mathscr{M}\!od_R)$, and it is evidently an abelian group. The zero object is the zero module. The direct sum of E and F consists of all pairs (x,y) with $x \in E$ and $y \in F$ with coordinatewise operations, the R-module maps μ and π are the inclusions and the projections into and onto the respective coordinate. Hence, the direct sum $E \oplus F$ is isomorphic to the direct product $E \times F$ (as is the case in any additive category, where the terminology *coproduct* is used instead of direct sum).

The canonical choice for the class $\mathscr{M}$ is the one consisting of all monomorphisms in $\mathscr{M}\!od_R$; these agree with the one-to-one R-module maps. The category $\mathscr{M}\!od_R$ has enough $\mathscr{M}$-injectives [10, Proposition I.8.3].

Example 3.3. Since the sum of two contractions is not a contraction, the category $\mathscr{B}an^1$ is not additive. However, the larger category $\mathscr{B}an^\infty$ is: the sum of two bounded linear operators is bounded, the zero object is the zero Banach space, and the direct sum of two Banach spaces exists. In this case, for $E, F \in \operatorname{obj}(\mathscr{B}an^\infty)$, $\operatorname{Mor}_{\mathscr{B}an^\infty}(E,F)$ is typically written as $\mathscr{L}(E,F)$ and is another object in $\mathscr{B}an^\infty$ (a difference to $\mathscr{M}od_R$). The monomorphisms in $\mathscr{B}an^\infty$ are the one-to-one bounded operators and the epimorphisms those with dense range. Thus, for $f \in \mathscr{L}(E,F)$ to be an isomorphism in $\mathscr{B}an^\infty$ it is not sufficient to be both a monomorphism and an epimorphism.

For the class $\mathscr{M}$ one could take the same as in $\mathscr{B}an^1$; but then not all isomorphisms (bijective bounded operators) would be in $\mathscr{M}$. So the canonical choice is the one-to-one bounded operators with closed range. By Example 2.2, $\mathscr{B}an^\infty$ has enough $\mathscr{M}$-injectives. Let us point out a subtle difference in the notions of injectivity in $\mathscr{B}an^1$ and in $\mathscr{B}an^\infty$. A Banach space I which is injective in $\mathscr{B}an^1$ is also injective in $\mathscr{B}an^\infty$: see the normalisation argument in Example 2.1. But if I is injective in $\mathscr{B}an^\infty$ it need not be injective in $\mathscr{B}an^1$ as the extension may not preserve the norm.

There is a neat way to describe injectivity in a category by 'comparison' with the category $\mathscr{A}b$ of abelian groups with group homomorphisms; this is done via the concept of an 'exact functor'. To introduce this notion, we firstly look at module categories. A sequence in $\mathscr{M}od_R$,

$$0 \longrightarrow E \xrightarrow{\mu} F \xrightarrow{\pi} G \longrightarrow 0 \tag{3.1}$$

is called *short exact* if μ is a monomorphism (one-to-one), π is an epimorphism (onto) and the image of μ agrees with the kernel of π. We introduce the *contravariant* Hom-*functor* as follows. Let $I \in \operatorname{obj}(\mathscr{M}od_R)$ be arbitrary and define

$$\begin{aligned} \operatorname{Hom}_R(-,I)\colon \mathscr{M}od_R &\longrightarrow \mathscr{A}b \\ E &\longmapsto \operatorname{Hom}_R(E,I) \\ \operatorname{Hom}_R(E,G) \ni f &\longmapsto f^* = \operatorname{Hom}_R(f,I) \end{aligned} \tag{3.2}$$

given by $f^*(g) = gf$ for $g \in \operatorname{Hom}_R(G,I)$. Then $f^*\colon \operatorname{Hom}_R(G,I) \to \operatorname{Hom}_R(E,I)$ is a group homomorphism, and 'contravariant' means that $(f_1 f_2)^* = f_2^* f_1^*$ for composable morphisms f_1 and f_2.

It is easy to check that this functor turns the sequence (3.1) above into the sequence

$$0 \longrightarrow \operatorname{Hom}_R(G,I) \xrightarrow{\pi^*} \operatorname{Hom}_R(F,I) \xrightarrow{\mu^*} \operatorname{Hom}_R(E,I) \tag{3.3}$$

where π^* is one-to-one and the image of π^* equals the kernel of μ^* but μ^* need not be surjective. One says the functor $\operatorname{Hom}_R(-,I)$ is *left exact*. In the case that μ^* *is* surjective—so that (3.3) turns into an exact sequence in $\mathscr{A}b$—one calls the functor *exact*.

With $\mathscr{M}$ still the class of all monomorphisms in $\mathscr{M}od_R$ we find that $I \in \operatorname{obj}(\mathscr{M}od_R)$ is $\mathscr{M}$-injective if and only if the functor $\operatorname{Hom}_R(-,I)$ is exact. The idea behind using a functor is that properties in the image category, such as $\mathscr{A}b$ for example, may be easier to understand.

Before moving on to more general categories, we wish to make the following point. A morphism $\mu \in \operatorname{Hom}_R(E,F)$ is always 'the first half' of a short exact sequence as in (3.1): we only have to take for G the quotient $F/\operatorname{im}\mu$, where $\operatorname{im}\mu$ is the image of μ, and π the canonical quotient mapping. This point of view will be stressed very soon below.

Let $\mathscr{A}$ be an additive category and let $f \in \operatorname{Mor}_{\mathscr{A}}(E,F)$ for some $E, F \in \operatorname{obj}(\mathscr{A})$.

Definition 3.4. A morphism $i\colon K \to E$ is a *kernel of* f if $fi = 0$ and for each $D \in \operatorname{obj}(\mathscr{A})$ and $g \in \operatorname{Mor}_{\mathscr{A}}(D,E)$ with $fg = 0$ there is a unique $h \in \operatorname{Mor}_{\mathscr{A}}(D,K)$

making the diagram below commutative

$$\begin{tikzcd} & D \arrow[dl, dashed, "h"'] \arrow[d, "g"] \arrow[dr, bend left, "0"] & \\ K \arrow[r, "i"] \arrow[rr, bend right, "0"'] & E \arrow[r, "f"] & F \end{tikzcd} \tag{3.4}$$

Any kernel is a monomorphism and is, up to isomorphism, unique. Thus we shall write $i = \ker f$.

Definition 3.5. A morphism $p\colon F \to C$ is a *cokernel of* f if $pf = 0$ and for each $D \in \operatorname{obj}(\mathscr{A})$ and $g \in \operatorname{Mor}_{\mathscr{A}}(F, D)$ with $gf = 0$ there is a unique $h \in \operatorname{Mor}_{\mathscr{A}}(C, D)$ making the diagram below commutative

$$\begin{tikzcd} E \arrow[r, "f"'] \arrow[rr, bend left, "0"] \arrow[dr, bend right, "0"'] & F \arrow[r, "p"'] \arrow[d, "g"] & C \arrow[dl, dashed, "h"] \\ & D & \end{tikzcd} \tag{3.5}$$

Any cokernel is an epimorphism and is, up to isomorphism, unique. Thus we shall write $p = \operatorname{coker} f$.

Example 3.6. Let E, F be Banach spaces and let $f \in \mathscr{L}(E, F)$ be a bounded linear operator. A kernel of f is the isometric embedding of $\ker f = \{x \in E \mid f(x) = 0\}$ into E. A cokernel of f is the open quotient mapping $F \mapsto F/\overline{\operatorname{im} f}$, where $\overline{\operatorname{im} f}$ stands for the closure of the subspace $\operatorname{im} f = \{f(x) \mid x \in E\}$.

Since the composition of a kernel with an isomorphism is a kernel, a monomorphism in $\mathscr{Ban}^\infty$ is a kernel if and only if it has closed image (by the Open Mapping Theorem). Likewise, an epimorphism is a cokernel if and only if it is surjective.

Let

$$\ell^1 = \bigl\{(\xi_n)_{n\in\mathbb{N}} \mid \sum_{n=1}^{\infty} |\xi_n| < \infty\bigr\}$$

be the space of all absolutely summable complex sequences with its canonical norm and let $c_0 = c_0(\mathbb{N})$. Then the embedding $\ell^1 \hookrightarrow c_0$ is both a monomorphism and an epimorphism but neither a kernel, nor a cokernel, nor an isomorphism.

Good sources of information on categories of Banach spaces are, e.g., [6, Chapter IV] and [7].

It turns out that the correct generalisation of short exact sequences in general additive categories is the concept of 'kernel–cokernel pairs'.

Definition 3.7. In an additive category $\mathscr{A}$, a *kernel–cokernel pair* (μ, π) consists of two composable morphisms in $\mathscr{A}$ such that $\mu = \ker \pi$ and $\pi = \operatorname{coker} \mu$, depicted as

$$\begin{tikzcd} E_1 \arrow[r, tail, "\mu"] & E_2 \arrow[r, two heads, "\pi"] & E_3 \end{tikzcd} \tag{3.6}$$

where $E_i \in \operatorname{obj}(\mathscr{A})$. A monomorphism arising in such a pair is called *admissible* and is denoted as

$$\begin{tikzcd} E \arrow[r, tail] & F \end{tikzcd}$$

and an epimorphism arising in such a pair is called *admissible* and is denoted as

$$\begin{tikzcd} E \arrow[r, two heads] & F \end{tikzcd}$$

Evidently this is a generalisation of (3.1) in $\mathscr{M}od_R$. The categories that resemble module categories most are the abelian categories which are now discussed in the next section.

4. Abelian vs. Exact Categories

One of the main technical devices in Homological Algebra are the 'diagram lemmas' which allow for (often skillful) manipulations with morphisms. In order for these to be possible one often requires the additive category $\mathscr{A}$ to satisfy two further conditions

(i) every morphism in $\mathscr{A}$ has both a kernel and a cokernel;
(ii) every monomorphism is a kernel and every epimorphism is a cokernel.

In this case, $\mathscr{A}$ is an *abelian* category. These seemingly innocent looking additional requirements have far-reaching consequences. For example, it follows that every morphism which is both a monomorphism and an epimorphism is already an isomorphism. In addition, every morphism f can be uniquely factorised as

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi"'] & & F \\ & G \arrow[ur, "\mu"'] & \end{tikzcd} \tag{4.1}$$

where π is an epimorphism and μ is a monomorphism. Clearly, $\mathscr{M}od_R$ is an abelian category and, in fact, every abelian category can, in some sense, be 'embedded' into a module category (the Freyd–Mitchell embedding theorem [14, Section VI.7]). The short exact sequences can then equivalently be expressed by (3.6).

Alas, the categories in functional analysis such as $\mathscr{B}an^\infty$ are typically not abelian, see Example 3.6. Among the many generalisations of abelian categories the one that seems to work best for us is the concept of an exact category in the sense of Quillen; see [5] and [6].

Definition 4.1. An *exact structure* on an additive category $\mathscr{A}$ is a class of kernel–cokernel pairs, closed under isomorphisms, satisfying the following axioms.

[E0] $\forall\ E \in \mathscr{A}\colon \mathrm{id}_E$ is an admissible monomorphism;
[E0$^{\mathrm{op}}$] $\forall\ E \in \mathscr{A}\colon \mathrm{id}_E$ is an admissible epimorphism;
[E1] the class $\mathscr{M}$ of admissible monomorphisms is closed under composition;
[E1$^{\mathrm{op}}$] the class $\mathscr{P}$ of admissible epimorphisms is closed under composition;
[E2] the push-out of an admissible monomorphism along an arbitrary morphism exists and yields an admissible monomorphism;
[E2$^{\mathrm{op}}$] the pull-back of an admissible epimorphism along an arbitrary morphism exists and yields an admissible epimorphism.

Together with an exact structure, $\mathscr{A}$ is called an *exact category.* We will also use the notation $\mathscr{E} = (\mathscr{M}, \mathscr{P})$ to denote an exact structure.

It is not a coincidence that we chose the symbol $\mathscr{M}$ above; this will become clear in the next section. An easy exercise shows that an abelian category equipped with the exact structure given by all monomorphisms and all epimorphisms is an exact category. On the other hand, $\mathscr{B}an^\infty$ is a non-abelian category which is an exact category when endowed with the structure $\mathscr{E}_{\max}$ of *all* kernel–cokernel pairs, see [6, Theorem 2.3.3].

We can now make contact with the notion of retract introduced in Section 2, Definition 2.3.

Definition 4.2. A kernel–cokernel pair in an exact category $\mathscr{A}$,

$$E \rightarrowtail^{\mu} F \xrightarrow{\pi} \twoheadrightarrow G$$

is *split* if there exist morphisms $\nu \in \mathscr{M}(F, E)$ and $\iota \in \mathscr{P}(G, F)$ that make F a direct sum of E and G (where $\mathscr{P}(G, F) = \{\rho \in \mathrm{Mor}_{\mathscr{A}}(G, F) \mid \rho \in \mathscr{P}\}$).

The following result is the analogue of the 'Splitting Lemma' in module theory.

Proposition 4.3. *Let $\mathscr{A}$ be an exact category. The following are equivalent for a kernel-cokernel pair* $E \overset{\mu}{\rightarrowtail} F \overset{\pi}{\twoheadrightarrow} G$ *in* $(\mathscr{M}, \mathscr{P})$*:*
(a) *The kernel–cokernel pair is split;*
(b) *E is a retract of F with section μ;*
(c) *G is a retract of F with retraction π.*

Proof. By definition, (a) implies both (b) and (c). Assume (b) and let $\nu \in \mathrm{Hom}_{\mathscr{A}}(F, E)$ be such that $\nu\mu = \mathrm{id}_E$. Then $(\mathrm{id}_F - \mu\nu)\mu = 0$ so by the property of $\pi = \operatorname{coker}\mu$ there is $\iota \in \mathrm{Hom}_{\mathscr{A}}(G, F)$ such that $\mathrm{id}_F - \mu\nu = \iota\pi$ and hence, $\mathrm{id}_F = \mu\nu + \iota\pi$. Moreover,

$$\pi\iota\pi = \pi(\mathrm{id}_F - \mu\nu) = \pi - \pi\mu\nu = \pi$$

so that $\pi\iota = \mathrm{id}_G$ follows as π is an epimorphism.

The implication (c) $\Rightarrow$ (a) is proved in a similar way. □

In analogy with module theory we introduce the following concept.

Definition 4.4. An object $F \in \mathrm{obj}(\mathscr{A})$ is called *$\mathscr{M}$-semisimple* if all kernel–cokernel pairs of the form $E \overset{\mu}{\rightarrowtail} F \overset{\pi}{\twoheadrightarrow} G$ in $(\mathscr{M}, \mathscr{P})$ split.

Corollary 4.5. *The following are equivalent:*
(a) *Every object in $\mathscr{A}$ is $\mathscr{M}$-injective;*
(b) *Every kernel–cokernel pair in $(\mathscr{M}, \mathscr{P})$ is split;*
(c) *Every object in $\mathscr{A}$ is $\mathscr{M}$-semisimple.*

Proof. This follows immediately from the definitions, Proposition 4.3 and Proposition 2.4. □

Example 4.6. Let R be a unital ring and let $\mathscr{A} = \mathscr{M}od_R$. Let $\mathscr{M}$ be the class of all monomorphisms in $\mathscr{A}$. Then $\mathscr{M}$-injectivity is the usual injectivity considered in module theory, and the statement in Corollary 4.5 above is well known. In addition, see, e.g., [15, Theorem 4.40], every right R-module is projective; every right R-module is a direct sum of simple submodules; and R is a finite direct product of matrix rings over division rings (the Artin–Wedderburn theorem). In this situation, R is termed *semisimple*.

5. Dimension

In this section we come back to the topic of dimension. Let us approach it from the point of view of splitting the short exact sequence (3.1):

$$0 \longrightarrow E \xrightarrow{\mu} F \xrightarrow{\pi} G \longrightarrow 0 \tag{5.1}$$

If G is a free module then π automatically is a retraction; this continues to hold if G is merely projective (a direct summand of a free module). On the other hand, if E is injective, then it is an absolute retract (Proposition 2.4) so μ is a section and the sequence splits too. We have a left–right symmetric situation here and it may thus not come as a surprise that, in module theory, the 'global dimension' of the ring R can be defined equivalently using projective or using injective modules; see, e.g., [15]. In other categories, for example sheaves of modules over ringed spaces or their analogues in C^*-theory, see [3], there are enough injective but not enough projective objects. This is why it may be desirable to focus on injectivity.

The starting point is: if an object is injective, its dimension should be 0. Now, and from now on, suppose we have enough injectives. Then any object can be embedded into an injective one and if it is a retract, then it is itself injective (Proposition 2.4) so the dimension is still 0. But if it is not a retract then its dimension should be at least 1. In this case it makes sense to consider the 'quotient' of the bigger injective object by the smaller non-injective one: if this turns out to be injective, one would say the dimension is equal to 1; otherwise at least 2. And so on ...

Let us formalise this process. Suppose $\mathscr{A}$ is an exact category and $\mathscr{M}$ the class of admissible monomorphisms (kernels of cokernels), cf. Definition 3.7. Take $E \in \mathrm{obj}(\mathscr{A})$. As $\mathscr{A}$ has enough $\mathscr{M}$-injectives, there are an $\mathscr{M}$-injective I^0 and $\mu \in \mathscr{M}(E, I^0)$. If μ is a section we are done. Otherwise let $\pi^0 = \operatorname{coker} \mu$ with codomain C^1. If C^1 is injective we stop. Otherwise there are an $\mathscr{M}$-injective I^1 and $\mu^0 \in \mathscr{M}(C^1, I^1)$. If μ^0 is a section we stop; and so on ...

$$
\begin{tikzcd}
E \arrow[r, tail, "\mu"] & I^0 \arrow[dr, two heads, "\pi^0"] & & & & \\
& & C^1 \arrow[r, tail, "\mu^0"] & I^1 \arrow[dr, two heads, "\pi^1"] & & \\
& & & & C^2 \arrow[r, tail, "\mu^1"] & I^2 \arrow[dr, dotted, two heads] & \\
& & & & & & {}
\end{tikzcd}
\tag{5.2}
$$

Why have we written this long sequence as a staircase? Note that (μ, π^0), (μ^0, π^1), ..., in general, (μ^{k-1}, π^k) are kernel–cokernel pairs while the morphisms $\mu^k \pi^k$ between I^{k-1} and I^k are compositions of a morphism in $\mathscr{P}$ followed by a morphism in $\mathscr{M}$. This means the sequence is 'exact' at the I^k whereas the morphisms between the $\mathscr{M}$-injective objects are of a special form.

Let's have a look again at the canonical factorisation of a morphism in an abelian category as displayed in (4.1). This is an essential ingredient in the workings of Homological Algebra; however, not every morphism in an exact category can be factorised in such a way. In fact, *if* every morphism can be factorised as in (4.1) in an exact category, then the category is already abelian. So we have to specialise to those morphisms, which is done below. In addition, we have to define 'long exact sequences'.

Definition 5.1. Let $\mathscr{A}$ be an exact category with exact structure $\mathscr{E} = (\mathscr{M}, \mathscr{P})$. The morphism $f \in \mathrm{Hom}_{\mathscr{A}}(E, F)$, $E, F \in \mathrm{obj}(\mathscr{A})$ is called *admissible* if it can be factorised as

$$
\begin{tikzcd}
E \arrow[rr, "f"] \arrow[dr, two heads, "\pi"'] & & F \\
& G \arrow[ur, tail, "\mu"'] &
\end{tikzcd}
$$

for some admissible monomorphism μ and some admissible epimorphism π in $\mathscr{A}$.

A sequence of admissible morphisms in $\mathscr{A}$,

$$
\begin{tikzcd}
E_1 \arrow[rr, "f_1"] \arrow[dr, two heads, "\pi_1"'] & & E_2 \arrow[rr, "f_2"] \arrow[dr, two heads, "\pi_2"'] & & E_3 \\
& G_1 \arrow[ur, tail, "\mu_1"'] & & G_2 \arrow[ur, tail, "\mu_2"'] &
\end{tikzcd}
$$

is said to be *exact* if the short sequence $G_1 \rightarrowtail^{\mu_1} E_2 \xrightarrow{\pi_2} \mathfrak{G}_2$ is exact (that is, $(\mu_1, \pi_2) \in (\mathscr{M}, \mathscr{P})$). An arbitrary sequence of admissible morphisms in $\mathscr{A}$ is *exact* if the sequences given by any two consecutive morphisms are exact.

We can now reformulate the above 'staircase' (5.2) by 'straightening it out' as follows.

Definition 5.2. Let $E \in \mathrm{obj}(\mathscr{A})$ for an exact category $\mathscr{A}$. An *$\mathscr{M}$-injective resolution* of E is a sequence of the form

$$E \rightarrowtail I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \dots \tag{5.3}$$

where all the morphisms d^k are admissible, the sequence is exact (at all I^k) and all I^k are $\mathscr{M}$-injective. (Note that this in particular implies that the sequence is a complex, that is, $d^k d^{k-1} = 0$ for all $k \in \mathbb{N}$.)

We are now in a position to define a dimension using injectivity.

Definition 5.3. Let $E \in \mathrm{obj}(\mathscr{A})$ for an exact category $\mathscr{A}$. We say E *has finite $\mathscr{M}$-injective dimension* if there exists a finite $\mathscr{M}$-injective resolution (5.3) such that d^{k-1} is a section for some $k \in \mathbb{N}$. In this case we define

$$\mathscr{M}\text{-}\dim(E) = \min\{k \in \mathbb{N} \mid d^{k-1} \text{ is a section}\} \tag{5.4}$$

as the *$\mathscr{M}$-injective dimension* of E. In case E does not have a finite $\mathscr{M}$-injective resolution we put $\mathscr{M}\text{-}\dim(E) = \infty$ and say that E has *infinite $\mathscr{M}$-injective dimension.*

Let us return to the staircase (5.2) using the same notation and put $d^{k-1} = \mu^{k-1}\pi^{k-1}$ for all $k \geq 1$ to obtain (5.3). Suppose d^{k-1} is a section with retraction ρ^{k-1} in $\mathrm{Hom}_{\mathscr{A}}(I^k, I^{k-1})$. From $\mathrm{id}_{I^{k-1}} = \rho^{k-1}d^{k-1} = \rho^{k-1}\mu^{k-1}\pi^{k-1}$ we obtain $\mathrm{id}_{G^k}\pi^{k-1} = \pi^{k-1}\rho^{k-1}\mu^{k-1}\pi^{k-1}$ which implies that $\mathrm{id}_{G^k} = \pi^{k-1}\rho^{k-1}\mu^{k-1}$ as π^{k-1} is an epimorphism. Hence μ^{k-1} is a section and G^k is a retract of the $\mathscr{M}$-injective object I^k, thus $\mathscr{M}$-injective by Proposition 2.4. Conversely, if G^k is $\mathscr{M}$-injective, then μ^{k-1} is a section (as G^k is an absolute $\mathscr{M}$-retract) and we can replace I^k by G^k. Therefore finite $\mathscr{M}$-injective dimension really determines the first $k \geq 0$ such that a morphism in an injective resolution is a cokernel just as intended in the explanation of the staircase. (Note also that, by Proposition 4.3, μ^{k-1} is a section if and only if π^k is a retraction.)

It would, however, be tedious to work through all possible injective resolutions in order to find the injective dimension of an object. This is where the Hom-functor comes in.

In the sequel, $\mathscr{A}$ will always denote an exact category with exact structure $(\mathscr{M}, \mathscr{P})$ and with enough injectives. Firstly we observe that *every* object $E \in \mathrm{obj}(\mathscr{A})$ has an $\mathscr{M}$-injective resolution; this is the construction in the staircase (5.2). Secondly, all such resolutions are equivalent in the following sense.

Definition 5.4. A *complex in $\mathscr{A}$*, denoted by $(E^\bullet, d^\bullet)$, is a sequence

$$\dots \longrightarrow E^{n-1} \xrightarrow{d^{n-1}} E^n \xrightarrow{d^n} E^{n+1} \longrightarrow \dots \tag{5.5}$$

such that $(E^n)_{n\in\mathbb{Z}}$ is a sequence of objects in $\mathscr{A}$, $(d^n)_{n\in\mathbb{Z}}$ is a sequence of admissible morphisms $d^n \in \mathrm{Hom}_{\mathscr{A}}(E^n, E^{n+1})$ and $d^{n+1}d^n = 0$ for all $n \in \mathbb{Z}$.

Let $(E^\bullet, d^\bullet)$ and $(F^\bullet, \partial^\bullet)$ be two complexes in $\mathscr{A}$. A *morphism* from $(E^\bullet, d^\bullet)$ to $(F^\bullet, \partial^\bullet)$ is a sequence of morphisms $E^n \to F^n$, $n \in \mathbb{Z}$ making the diagram below commutative

$$\begin{tikzcd} \dots \arrow[r] & E^{n-1} \arrow[r, "d^{n-1}"] \arrow[d] & E^n \arrow[r, "d^n"] \arrow[d] & E^{n+1} \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & F^{n-1} \arrow[r, "\partial^{n-1}"] & F^n \arrow[r, "\partial^n"] & F^{n+1} \arrow[r] & \dots \end{tikzcd} \tag{5.6}$$

Definition 5.5. Let $\varphi, \psi\colon (E^\bullet, d^\bullet) \to (F^\bullet, \partial^\bullet)$ be morphisms of complexes in $\mathscr{A}$. Then φ is *homotopic to* ψ, written as $\varphi \simeq \psi$, if there is a sequence $(\sigma^n)_{n\in\mathbb{Z}}$ of morphisms $\sigma^n \in \mathrm{Hom}_{\mathscr{A}}(E^n, F^{n-1})$ such that, for all $n \in \mathbb{Z}$, we have

$$\varphi_n - \psi_n = \partial^{n-1}\sigma^n + \sigma^{n+1}d^n. \tag{5.7}$$

This defines an equivalence relation on the class of morphisms of complexes.

The two complexes $(E^\bullet, d^\bullet)$ and $(F^\bullet, \partial^\bullet)$ are called *homotopic* if there exist morphisms $\varphi\colon (E^\bullet, d^\bullet) \to (F^\bullet, \partial^\bullet)$ and $\bar\varphi\colon (F^\bullet, \partial^\bullet) \to (E^\bullet, d^\bullet)$ such that $\bar\varphi\varphi \simeq \mathrm{id}_{(E^\bullet,d^\bullet)}$ and $\varphi\bar\varphi \simeq \mathrm{id}_{(F^\bullet,\partial^\bullet)}$.

To an $\mathscr{M}$-injective resolution (5.3) one associates a complex $(I^\bullet, d^\bullet)$, where E is deleted from the sequence and all $I^n = 0$ for $n < 0$ (in particular, $d^{-1} = 0$). The following is a standard result in Homological Algebra, see, e.g., [10, Proposition IV.4.5], since the arguments used in abelian categories take over in exact categories, cf. [5] and [16, Chapter 3].

Proposition 5.6. *Any two $\mathscr{M}$-injective resolutions of $E \in \mathrm{obj}(\mathscr{A})$ are homotopic.*

In analogy to the contravariant Hom-functor (3.2) one has the covariant Hom-functor. Let $F \in \mathrm{obj}(\mathscr{A})$ be arbitrary and define

$$\begin{aligned} \mathrm{Hom}_{\mathscr{A}}(F,-)\colon \mathscr{A} &\longrightarrow \mathscr{Ab} \\ E &\longmapsto \mathrm{Hom}_{\mathscr{A}}(F,E) \\ \mathrm{Hom}_{\mathscr{A}}(G,E) \ni f &\longmapsto f_* = \mathrm{Hom}(f,F) \end{aligned} \tag{5.8}$$

given by $f_*(g) = fg$ for $g \in \mathrm{Hom}_{\mathscr{A}}(F,G)$. Then $f_*\colon \mathrm{Hom}_{\mathscr{A}}(F,G) \to \mathrm{Hom}_{\mathscr{A}}(F,E)$ is a group homomorphism, and 'covariant' means that $(f_1f_2)^* = f_1^*f_2^*$ for composable morphisms f_1 and f_2.

Apply this functor to the complex $(I^\bullet, d^\bullet)$ to obtain a complex as below in $\mathscr{Ab}$

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{A}}(F,I^0) \xrightarrow{d_*^0} \mathrm{Hom}_{\mathscr{A}}(F,I^1) \xrightarrow{d_*^1} \mathrm{Hom}_{\mathscr{A}}(F,I^2) \longrightarrow \dots \tag{5.9}$$

In general, this is no longer an exact sequence so one applies homology, that is, takes the quotient group $\ker d_*^{k+1}/\operatorname{im} d_*^k$ which is possible since $d_*^{k+1}d_*^k = 0$.

Definition 5.7. Let $\mathscr{A}$ be an exact category with enough injectives. Let $F \in \mathrm{obj}(\mathscr{A})$ be fixed. Let $E \in \mathrm{obj}(\mathscr{A})$ and $(I^\bullet, d^\bullet)$ be the complex associated to an $\mathscr{M}$-injective resolution (5.3) of E. Each $\ker d_*^{k+1}/\operatorname{im} d_*^k$ is called the *k-th cohomology group* and will be denoted by $\mathrm{Ext}^k(F,E)$.

Remark 5.8. Either by definition or left exactness of $\mathrm{Hom}_{\mathscr{A}}(F,-)$ we have $\mathrm{Ext}^0(F,E) \cong \mathrm{Hom}_{\mathscr{A}}(F,E)$ for all F and E.

Though it appears that the above definition depends on the choice of the injective resolution, in fact, by Proposition 5.6, any two injective resolutions of E are homotopic and this is preserved by the functor $\mathrm{Hom}_{\mathscr{A}}(F,-)$. As a consequence, the homology is the same. For details, see, e.g., [10, Section IV.3]. Moreover, for each $\varphi \in \mathrm{Hom}_{\mathscr{A}}(E,E')$ one can define a homomorphism $\varphi_*\colon \mathrm{Ext}^k(F,E) \to \mathrm{Ext}^k(F,E')$, $k \in \mathbb{N}$ and hence obtains the *k-th right derived functor of* $\mathrm{Hom}_{\mathscr{A}}(F,-)$. See [10, Section IV.5] for more details.

We finally state how these gadgets can help to determine the injective dimension.

5.9 Injective Dimension Theorem. *Let $\mathscr{A}$ be an exact category with enough injectives. Let $n \in \mathbb{N}$. The following are equivalent for an object $E \in \mathrm{obj}(\mathscr{A})$:*

(a) $\mathscr{M}\text{-}\dim(E) \le n$;
(b) $\mathrm{Ext}^m(F,E) = 0$ *for all* $m > n$ *and all* $F \in \mathrm{obj}(\mathscr{A})$;
(c) $\mathrm{Ext}^{n+1}(F,E) = 0$ *for all* $F \in \mathrm{obj}(\mathscr{A})$;

(d) $\operatorname{Ext}^n(-,E)\colon \mathscr{A} \to \mathscr{Ab}$ *is exact;*
(e) *there exists an* $\mathscr{M}$*-injective resolution of* E *whose* n*-th cokernel is* $\mathscr{M}$*-injective;*
(f) *for every* $\mathscr{M}$*-injective resolution of* E *the* n*-th cokernel is* $\mathscr{M}$*-injective.*

A proof of this result for module categories and more general abelian categories can be found in [15] and its extension to exact categories in [16].

6. Operator Modules

In [16], the above theory is applied to the category of operator modules over a C^*-algebra and, in more general form, to sheaves of operator modules over C^*-ringed spaces in [3]; see also [13]. Throughout this section A will denote a unital C^*-algebra.

Definition 6.1. A unital right A-module E which at the same time is an operator space is a *right operator* A*-module* if it satisfies either of the following equivalent conditions:
(a) There exist a complete isometry $\Phi\colon E \longrightarrow B(H,K)$, for some Hilbert spaces H, K, and a *-homomorphism $\pi\colon A \longrightarrow B(H)$ such that $\Phi(x\cdot a) = \Phi(x)\pi(a)$ for all $x \in E$, $a \in A$.
(b) The bilinear mapping $E \times A \longrightarrow E$, $(x,a) \mapsto x\cdot a$ extends to a complete contraction $E \otimes_h A \longrightarrow E$.
(c) For each $n \in \mathbb{N}$, $M_n(E)$ is a right Banach $M_n(A)$-module in the canonical way.

Our general reference for operator modules is [4], where, for instance, the Haagerup tensor product in the above definition, part (b) is treated in great detail. See also [16, Appendix A] for an in-depth discussion of this type of module and comparisons to other kinds of 'operator space modules'. We will denote by $\mathscr{OMod}_A^\infty$ the category with objects the right operator A-modules and morphisms the completely bounded A-module maps. It is similar to the category $\mathscr{Ban}^\infty$, in particular it is not abelian, but the morphisms respect the so-called matricial structure of a C^*-algebra, which has become important in that area since the 1970s.

In $\mathscr{OMod}_A^\infty$, a morphism T is a kernel iff it is a completely bounded isomorphism onto its image, and it is a cokernel iff it is surjective and completely open. (Note that there is no Open Mapping Theorem for operator spaces.)

Theorem 6.2 ([16], Theorem 4.40; see also [3])**.** *The class* $(\mathscr{M},\mathscr{P})$ *of all kernel–cokernel pairs in* $\mathscr{OMod}_A^\infty$ *is an exact structure on* $\mathscr{OMod}_A^\infty$*.*

Consequently, and since $\mathscr{OMod}_A^\infty$ has enough injectives, by Wittstock's Hahn–Banach theorem [8, Theorem 4.1.5], we can apply the ideas developed above. One is particularly interested in an invariant for the C^*-algebra A, and hence defines a 'global dimension' in analogy to the concept from ring theory.

Definition 6.3. The *global* C^**-dimension* of a (unital) C^*-algebra A is defined by

$$C^*\text{-}\dim(A) = \sup\{\mathscr{M}\text{-}\dim(E) \mid E \in \mathscr{OMod}_A^\infty\}.$$

Recall, from Example 4.6, that a unital ring R is semisimple (in the classical sense) if and only if every module in $\mathscr{Mod}_R$ is injective; that is, has global dimension equal to zero. These rings are described by the Artin–Wedderburn theorem. One might hope that a similar class of C^*-algebras could also be identified; however, this is not the case!

Example 6.4. The unital C^*-algebra $\mathbb{C}$ has global C^*-dimension greater than 0. This follows immediately from the fact that c_0, viewed as a $\mathbb{C}$-module in a canonical way, is an operator module and is completely isometrically embedded into ℓ^∞. The latter is injective as an operator module (as every bounded linear map into ℓ^∞ is completely bounded [8, Proposition 2.2.6]) and thus, if c_0 was injective, it would have to be a retract of ℓ^∞ (Proposition 2.4) which it is not (Example 2.5). Thus c_0 is not $\mathscr{M}$-injective in $\mathscr{OMod}_{\mathbb{C}}^\infty$.

In fact, the same statement holds for every unital C^*-algebra A; one can use the compact operators on an infinite-dimensional Hilbert space in place of c_0. But let us move on to dimension 1.

Proposition 6.5. *The global C^*-dimension of A is at most one if and only if every complete quotient of an $\mathscr{M}$-injective object in $\mathcal{OMod}_A^\infty$ is $\mathscr{M}$-injective.*

This follows immediately from the Injective Dimension Theorem (5.9) as a complete quotient is nothing but the image F of a cokernel so we can apply the equivalence of (a), (e) and (f) to an injective presentation $E \overset{\mu}{\rightarrowtail} I \overset{\pi}{\twoheadrightarrow} F$ with I $\mathscr{M}$-injective.

But it turns out that the condition in the above proposition always fails.

Theorem 6.6. *The global C^*-dimension of every unital C^*-algebra is at least* 2.

The details of the proof can be found in [16, Chapter 5]; an important ingredient is the injective presentation $K(H) \rightarrowtail B(H) \twoheadrightarrow B(H)/K(H)$ for an infinite-dimensional Hilbert space H and the classical fact that ℓ^∞/c_0 is not injective in $\mathscr{Ban}^\infty$ [2].

At this moment, no C^*-algebra with global C^*-dimension equal to 2 is known; in fact, it is unclear whether there is any C^*-algebra with finite dimension.

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