Homogeneous manifolds whose geodesics are orbits. Recent results and some open problems

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ABSTRACT. A homogeneous Riemannian manifold $(M = G/K, g)$ is called a space with homogeneous geodesics or a G-g.o. space if every geodesic $\gamma(t)$ of M is an orbit of a one-parameter subgroup of G, that is $\gamma(t) = \exp(tX) \cdot o$, for some non zero vector X in the Lie algebra of G. We give an exposition on the subject, by presenting techniques that have been used so far and a wide selection of previous and recent results. We discuss generalization to two-step homogeneous geodesics. We also present some open problems.

1. INTRODUCTION

The aim of the present article is to give an exposition on developments about homogeneous geodesics in Riemannian homogeneous spaces, to present various recent results and discuss some open problems. One of the demanding problems in Riemannian geometry is the description of geodesics. By making some symmetry assumptions one could expect that certain simplifications may accur. Let (M, g) be a homogeneous Riemannian manifold, i.e. a connected Riemannian manifold on which the largest connected group G of isometries acts transitively. Then M can be expressed as a homogeneous space $(G/K, g)$, where K is the isotropy group at a fixed point o of M.

Motivated by well known facts such that, the geodesics in a Lie group G with a bi-invariant metric are the one-parameter subgroups

of G, or that the geodesics in a Riemannian symmetric space G/K are orbits of one-parameter subgroups in G/K, it is natural to search for geodesics in a homogeneous space, which are orbits. More precisely, a geodesic $\gamma(t)$ through the origin o of $M = G/K$ is called *homogeneous* if it is an orbit of a one-parameter subgroup of G, that is

$$\gamma(t) = \exp(tX) \cdot o, \quad t \in \mathbb{R}, \tag{1}$$

where X is a non zero vector in the Lie algebra $\mathfrak{g}$ of G. A non zero vector $X \in \mathfrak{g}$ is called a *geodesic vector* if the curve (1) is a geodesic. A homogeneous Riemannian manifold $M = G/K$ is called a *g.o. space* if all geodesics are homogeneous with respect to the largest connected group of isometries $I_o(M)$. Since their first systematic study by O. Kowalski and L. Vanhecke in [45], there has been a lot of research related to homogeneous geodesics and g.o spaces and in various directions.

Homogeneous geodesics appear quite often in physics as well. The equation of motion of many systems of classical mechanics reduces to the geodesic equation in an appropriate Riemannian manifold M. Homogeneous geodesics in M correspond to "relative equilibriums" of the corresponding system (cf. [6]). For further information about relative equilibria in physics we refer to [36] and references therein. In Lorentzian geometry in particular, homogeneous spaces with the property that all their *null* geodesics are homogeneous, are candidates for constructing solutions to the 11-dimensional supergravity, which preserve more than 24 of the available 32 supersymmetries. In fact, all Penrose limits, preserving the amount of supersymmetry of such a solution, must preserve homogeneity. This is the case for the Penrose limit of a reductive homogeneous spacetime along a null homogeneous geodesic ([35], [50], [55]). For a recent mathematical contribution in this topic see [28].

All naturally reductive spaces are g.o. spaces ([41]), but the converse is not true in general. In [39] A. Kaplan proved the existence of g.o. spaces that are in no way naturally reductive. These are generalized Heisenberg groups with two dimensional center. Another important class of g.o. spaces are the weakly symmetric spaces. These are homogeneous Riemannian manifolds $(M = G/K, g)$ introduced by A. Selberg in [57], with the property that every two points can be interchanged by an isometry of M. In [15] J. Berndt, O. Kowalski and L. Vanhecke proved that weakly symmetric spaces

are g.o. spaces. In [42] O. Kowalski, F. Prüfer and L. Vanhecke gave an explicit classification of all naturally reductive spaces up to dimension five, and in [1] the authors classified naturally reductive homogeneous spaces up to dimension six. The classification in dimensions seven and eight was recently completed ([58]).

The term *g.o. space* was introduced by O. Kowalski and L. Vanhecke in [45], where they gave the classification of all g.o. spaces up to dimension six, which are in no way naturally reductive. Concerning the existence of homogeneous geodesics in a homogeneous Riemannian manifold, we recall the following. In ([38]) V.V. Kajzer proved that a Lie group endowed with a left-invariant metric admits at least one homogeneous geodesic. O. Kowalski and J. Szenthe extended this result to all homogeneous Riemannian manifolds ([44]). An extension of this result to reductive homogeneous pseudo-Riemannian manifolds was obtained ([31], [55]).

In [37] C. Gordon described g.o. spaces which are nilmanifolds and in [63] H. Tamaru classified homogeneous g.o. spaces which are fibered over irreducible symmetric spaces. In [26] and [30] O. Kowalski and Z. Dušek investigated homogeneous geodesics in Heisenberg groups and some H-type groups. Examples of g.o. spaces in dimension seven were obtained by Dušek, O. Kowalski and S. Nikčević in [32].

In [3] the author and D.V. Alekseevsky classified generalized flag manifolds which are g.o. spaces. Further, D.V. Alekseevsky and Yu. G. Nikonorov in [4] studied the structure of compact g.o. spaces and gave some sufficient conditions for existence and non existence of an invariant metric with homogeneous geodesics on a homogeneous space of a compact Lie group. They also gave a classification of compact simply connected g.o. spaces of positive Euler characteristic.

In [40] O. Kowalski, S. Nikčević and Z. Vlášek studied homogeneous geodesics in homogneous Riemannian manifolds, and in [49], [20] G. Calvaruso and R. Marinosci studied homogeneous geodesics in three-dimensional Lie groups. Homogeneous geodesics were also studied by J. Szenthe in [59], [60], [61], [62]. Also, D. Latifi studied homogeneous geodesics in homogeneous Finsler spaces ([46]), and the first author investigated homogeneous geodesics in the flag manifold $SO(2l+1)/U(l-m) \times SO(2m+1)$ ([7]).

Homogeneous geodesics in the affine setting were studied in [26] and [33] (and in particular for any non reductive pseudo-Riemannian manifold).

Finally, a class of homogeneous spaces which satisfy the g.o. property are the δ-homogeneous spaces, which were introduced by V. Berestovskiĭ and C. Plaut in [14]. These spaces have interesting geometrical properties, but we will not persue here. We refer to the paper [13] by V. Berestovskiĭ and Yu.G. Nikonorov for more information in this direction. Further useful information about geodesic orbit spaces can be found in the recent work [53].

The paper is organized as follows. In Section 2 we present the basic techniques for finding homogeneous geodesics and detecting if a homogeneous space is a space with homogeneous geodesics (g.o. space). In Section 3 we present the classification up to dimension 6 and give examples in dimension 7. In Section 4 we discuss homogeneous g.o. spaces which are fibered over irreducible symmetric spaces and in Section 5 we present the classification of generalized flag manifolds which are g.o. spaces. In Section 6 we present results about another wide class of homogeneous spaces, the generalized Wallach spaces, and in Section 6 we discuss results related to M-spaces. These are homogeneous spaces G/K_1 so that $G/(S \times K_1)$ is a generalized flag manifold, where S a torus in a compact simple Lie group G. The pseudo-Riemannian setting is presented in Section 8. In Section 9 we discuss a generalization of homogeneous geodesics which we call two-step homogeneous geodesics. These are orbits of the product of two exponential factors. Finally, in Section 10 we present some open problems.

2. Homogeneous geodesics in homogeneous Riemannian manifolds - Techniques

A *homogeneous Riemannian manifold* is a Riemannian manifold M for which there exists a connected Lie group $G \subset I_0(M)$ which acts transitively on M as a group of isometries. Let $p \in M$ be a fixed point. If we denote by K the isotropy group at p, then M can be identified with the homogeneous space G/K. Note that there may exist more than one transitive isometry groups $G \subset I_0(M)$ so that M is represented as a coset space in more than one ways. For any fixed choice $M = G/K$, G acts effectively on G/K from the

left. A G-invariant metric g on $M = G/K$ is a Riemannian metric so that the diffeomorphism $p \mapsto a \cdot p$ is an isometry.

It is known ([41]) that a Riemannian homogeneous space is always *reductive*. This means that if $\mathfrak{g}$, $\mathfrak{k}$ are the Lie algebras of G and K respectively, then there is a direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \tag{2}$$

with $\mathrm{Ad}(K)(\mathfrak{m}) \subset \mathfrak{m}$. The canonical projection $\pi : G \to G/K$ induces an isomorphism between the subspace $\mathfrak{m}$ of $\mathfrak{g}$ and the tangent space $T_o M$ at the identity $o = eK$.

A G-invariant Riemannian metric g defines a scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$ which is $\mathrm{Ad}(K)$-invariant and vice-versa. If G is semisimple and compact and B denotes the negative of the Killing form of $\mathfrak{g}$, then any $\mathrm{Ad}(K)$-invariant scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$ can be expressed as $\langle x, y \rangle = B(\Lambda x, y)$ $(x, y \in \mathfrak{m})$, where Λ is an $\mathrm{Ad}(K)$-equivariant positive definite symmetric operator on $\mathfrak{m}$. Conversely, any such operator Λ determines an $\mathrm{Ad}(K)$-invariant scalar product $\langle x, y \rangle = B(\Lambda x, y)$ on $\mathfrak{m}$, which in turn determines a G-invariant Riemannian metric g on $\mathfrak{m}$. A Riemannian metric generated by the scalar product product B is called *standard metric*.

Definition 1. *A homogeneous Riemannian manifold $(M = G/K, g)$ is called a space with homogeneous geodesics, or G-g.o. space if every geodesic γ of M is an orbit of a one-parameter subgroup of G, that is $\gamma(t) = \exp(tX) \cdot o$, for some non zero vector $X \in \mathfrak{g}$. The invariant metric g is called G-g.o. metric. If G is the full isometry group, then the G-g.o. space is called a manifold with homogeneous geodesics, or a g.o. manifold.*

Notice that if all geodesics through the origin $o = eK$ are of the form $\gamma(t) = \exp(tX) \cdot o$, then the geodesics through any other point $a \cdot p$ $(a \in G, p \in M)$ is of the form $a\gamma(t) = \exp(t \, \mathrm{Ad}(a)X) \cdot (a \cdot p)$.

Definition 2. *A non zero vector $X \in \mathfrak{g}$ is called a geodesic vector if the curve (1) is a geodesic.*

All calculations for a g.o. space G/K can be reduced to algebraic computations using geodesic vectors. These can be computed by using the following fundamental result of the subject, still call it "lemma" by tradition:

Lemma 2.1 (Geodesic Lemma [45])**.** *A non zero vector $X \in \mathfrak{g}$ is a geodesic vector if and only if*

$$\langle [X, Y]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = 0, \tag{3}$$

for all $Y \in \mathfrak{m}$. Here the subscript $\mathfrak{m}$ denotes the projection into $\mathfrak{m}$.

A useful description of homogeneous geodesics (1) is provided by the following :

Proposition 2.2. ([3]) *Let $(M = G/K, g)$ be a homogeneous Riemannian manifold and Λ be the associated operator. Let $a \in \mathfrak{k}$ and $x \in \mathfrak{m}$. Then the following are equivalent:*

(1) The orbit $\gamma(t) = \exp t(a + x) \cdot o$ of the one-parameter subgroup $\exp t(a + x)$ through the point $o = eK$ is a geodesic of M.

(2) $[a + x, \Lambda x] \in \mathfrak{k}$.

(3) $\langle [a, x], y \rangle = \langle x, [x, y]_{\mathfrak{m}} \rangle$ for all $y \in \mathfrak{m}$.

(4) $\langle [a + x, y]_{\mathfrak{m}}, x \rangle = 0$ for all $y \in \mathfrak{m}$.

As a consequence, we obtain the following characterization of g.o. spaces:

Corollary 2.3 ([3]). *Let $(M = G/K, g)$ be a homogeneous Riemannian manifold. Then $(M = G/K, g)$ is a g.o. space if and only if for every $x \in \mathfrak{m}$ there exists an $a(x) \in \mathfrak{k}$ such that*

$$[a(x) + x, \Lambda x] \in \mathfrak{k}.$$

Therefore, the property of being a g.o. space G/K, depends only on the reductive decomposition and the G-invariant metric metric g on $\mathfrak{m}$. That is, if $(M = G/H, g)$ is a g.o. space, then any locally isomorphic homogeneous Riemannian space $(M = G/H, g')$ is a g.o. space. Also, a direct product of Riemannian manifolds is a manifold with homogeneous geodesics if and only if each factor is a manifold with homogeneous geodesics.

In order to find all homogeneous geodesics in a homogeneous Riemannian manifold $(M = G/K, g)$ it suffices to find a decomposition of the form (2) and look for geodesic vectors of the form

$$X = \sum_{i=1}^{s} x_i e_i + \sum_{j=1}^{l} a_j A_j. \tag{4}$$

Here $\{e_i : i = 1, 2, \ldots, s\}$ is a convenient basis of $\mathfrak{m}$ and $\{A_j : j = 1, 2, \ldots, l\}$ is a basis of $\mathfrak{k}$. By substituting $X = e_i$ $(i = 1, \ldots, s)$ into equation (3) we obtain a system of linear algebraic equations for the

variables x_i and a_j. The geodesic vectors correspond to those solutions for which $x_1, \ldots, x_s$ are not all equal to zero. For some applications of this method we refer to [40] and [49]. Also, $(M = G/K, g)$ is a g.o. space if and only if for every non zero s-tuple $(x_1, \ldots, x_s)$ there is an l-tuple $(a_1, \ldots, a_l)$ satisfying all quadratic equations. A useful technique used for the characterization of Riemannian g.o. spaces is based on the concept of the geodesic graph, originally introduced in [59]. We first need the following definition.

Definition 3. *A Riemannian homogeneous space $(G/K, g)$ is called naturally reductive if there exists a reductive decomposition (2) of $\mathfrak{g}$ such that*

$$\langle [X, Z]_{\mathfrak{m}}, Y \rangle + \langle X, [Z, Y]_{\mathfrak{m}} \rangle = 0, \quad \text{for all} \ \ X, Y, Z \in \mathfrak{m}. \quad (5)$$

It is well known that condition (5) implies that all geodesics in G/K are homogeneous (e.g. [54]).

Definition 4. *A homogeneous Riemannian manifold (M, g) is naturally reductive if there exists a transitive group G of isometries for which the correseponding Riemannian homogeneous space $(G/K, g)$ is naturally reductive in the sense of Definition 3.*

Therefore, it could be possible that a homogeneous space $M = G/K$ is not naturally reductive for some group $G \in I_0(M)$ (the connected component of the full isometry group of M), but it is naturally reductive if we write $M = G'/K'$ for some larger group of isometries $G' \subset I_0(M)$.

Let $(M = G/K, g)$ be a g.o. space and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be an $\mathrm{Ad}(K)$-invariant decomposition. Then

(1) There exists an $\mathrm{Ad}(H)$-equivariant map $\eta : \mathfrak{m} \to \mathfrak{k}$ (a *geodesic graph*) such that for any $X \in \mathfrak{m} \setminus \{0\}$, the curve $\exp t(X + \eta(X)) \cdot o$ is a geodesic.

(2) A geodesic graph is either linear (which is equivalent to natural reductivity with respect to some $\mathrm{Ad}(K)$-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}'$) or it is non differentiable at the origin o.

It can be shown ([43]) that a geodesic graph (for a g.o. space) is uniquely determined by fixing an $\mathrm{Ad}(H)$-invariant scalar product on $\mathfrak{k}$. Examples of g.o. spaces by using geodesic graphs are given in [29], [32], and [43]. Conversely, the property (1) implies that G/K is a g.o. space.

Another technique for producing g.o. metrics was given by C. Gordon as shown below:

Proposition 2.4. ([37], [63]) *Let G be a connected semisimple Lie group and $H \supset K$ be compact Lie subgroups in G. Let M_F and M_C be the tangent spaces of $F = H/K$ and $C = G/H$ respectively. Then the metric $g_{a,b} = aB \mid_{M_F} + bB \mid_{M_C}, (a, b \in \mathbb{R}^+)$ is a g.o. metric on G/K if and only if for any $v_F \in M_F$, $v_C \in M_C$ there exists $X \in \mathfrak{k}$ such that*

$$[X, v_F] = [X + v_F, v_C] = 0.$$

Actually, Gordon proved a more general result based on description of naturally reductive left-invariant metrics on compact Lie groups given by J.E. D'Atri and W. Ziller in [24].

3. Low dimensional examples

The problem of a complete classification of g.o. manifolds is open. Even the classification all g.o. metrics on a given Riemannian homogeneous space is not trivial (cf. for example [51]). A complete classification is known up to dimesion 6, given by O. Kowalski and L. Vanhecke:

Theorem 3.1. ([45]) 1) *All Riemannian g.o. spaces of dimension up to 4 are naturally reductive.*

2) *Every 5-dimensional Riemannian g.o. space is either naturally reductive, or of isotropy type* SU(2).

3) *Every 6-dimensional Riemannian g.o. space is either naturally reductive or one of the following:*

(a) *A two-step nilpotent Lie group with two-dimensional center, equipped with a left-invariant Riemannian metric such that the maximal connected isotropy subgroup is isomorphic to* SU(2) *or* U(2). *Corresponding g.o. metrics depend on three real parameters.*

(b) *The universal covering space of a homogeneous Riemannian manifold of the form* $(M = SO(5)/U(2), g)$ *or* $(M = SO(4, 1)/U(2), g)$, *where* SO(5) *or* SO(4, 1) *is the identity component of the full isometry group, respectively. In each case, all corresponding invariant metrics g.o. metrics g depend on two real parameters.*

As pointed out by the authors in [45, p. 190], the g.o. spaces (a) and (b) are *in no way naturally reductive* in the following sence: whatever the representation of (M, g) as a quotient of the form

G'/K', where G' is a connected transitive group of isometries of (M, g), and whatever is the $\mathrm{Ad}(K)$-invariant decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{m}'$, the curve $\gamma(t) = \exp(tX) \cdot o$ is never a geodesic (for any $X \in \mathfrak{m} \setminus \{0\}$).

The first 7-dimensional example of a g.o. manifold was given by C. Gordon in [37]. This is a nilmanifold (i.e. a connected Riemannian manifold admitting a transitive nilpotent group of isometries), and it was obtained under a general construction of g.o. nilmanifolds. It took some time until some more 7-dimensional examples were given. In [32] Z. Dušek, O. Kowalski and S. Nikčević gave families of 7-dimensional g.o. metrics. Their main result is the following:

Theorem 3.2. ([32]) *On the 7-dimensional homogeneous space G/K = (SO(5)×SO(2))/ U(2) (or G/H = (SO(4, 1)×SO(2))/ U(2)) there is a family $g_{p,q}$ of invariant metrics depending on two parameters p, q (where the pairs (p, q) fill in an open subset of the plane) such that each homogeneous Riemannian manifold $(G/H, g_{p,q})$ is a locally irreducible and not naturally reductive Riemannian g.o. manifold.*

4. Fibrations over symmetric spaces

In the work [63] H. Tamaru classified homogeneous spaces $M = G/K$ satisfying the following properties: (i) M is fibered over irreducuble symmetric spaces G/H and (ii) certain G-invariant metrics on M are G-g.o. metrics. More precisely, for G connected and semisimple, and H, K compact with $G \supset H \supset K$, he considered the fibration

$$F = H/K \to M = G/K \to B = G/H$$

and the G-invariant metrics $g_{a,b}$ on M determined by the scalar products

$$\langle \, , \, \rangle = aB|_{\mathfrak{f}} + bB|_{\mathfrak{b}}, \quad a, b > 0.$$

Here $\mathfrak{f}$ and $\mathfrak{b}$ are the tangent spaces of F and B respectively, so that the tangent space of M at the origin is identified with $\mathfrak{f} \oplus \mathfrak{b}$. By using results from polar representations, he classifed all triplets (G, H, K) so that the metrics $g_{a,b}$ are G-g.o. metrics. The triplets of Lie algebras $(\mathfrak{g}, \mathfrak{h}, \mathfrak{k})$ so that $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair and $(\mathfrak{g}, \mathfrak{k})$ corresponds to a G-g.o. space G/K, are listed in Table 1.

	$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{k}$
1	$\mathfrak{so}(2n+1), n \geq 2$	$\mathfrak{so}(2n)$	$\mathfrak{u}(n)$
2	$\mathfrak{so}(4n+1), n \geq 1$	$\mathfrak{so}(4n)$	$\mathfrak{su}(2n)$
3	$\mathfrak{so}(8)$	$\mathfrak{so}(7)$	$\mathfrak{g}_2$
4	$\mathfrak{so}(9)$	$\mathfrak{so}(8)$	$\mathfrak{so}(7)$
5	$\mathfrak{su}(n+1), n \geq 2$	$\mathfrak{u}(n)$	$\mathfrak{su}(n)$
6	$\mathfrak{su}(2n+1), n \geq 2$	$\mathfrak{u}(2n)$	$\mathfrak{u}(1) \oplus \mathfrak{sp}(n)$
7	$\mathfrak{su}(2n+1), n \geq 2$	$\mathfrak{u}(2n)$	$\mathfrak{sp}(n)$
8	$\mathfrak{sp}(n+1), n \geq 1$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$	$\mathfrak{u}(1) \oplus \mathfrak{sp}(n)$
9	$\mathfrak{sp}(n+1), n \geq 1$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$	$\mathfrak{sp}(n)$
10	$\mathfrak{su}(2r+n), r \geq 2, n \geq 1$	$\mathfrak{su}(r) \oplus \mathfrak{su}(r+n) \oplus \mathbb{R}$	$\mathfrak{su}(r) \oplus \mathfrak{su}(r+n)$
11	$\mathfrak{so}(4n+2), n \geq 2$	$\mathfrak{u}(2n+1)$	$\mathfrak{su}(2n+1)$
12	$\mathfrak{e}_6$	$\mathbb{R} \oplus \mathfrak{so}(10)$	$\mathfrak{so}(10)$
13	$\mathfrak{so}(9)$	$\mathfrak{so}(7) \oplus \mathfrak{so}(2)$	$\mathfrak{g}_2 \oplus \mathfrak{so}(2)$
14	$\mathfrak{so}(10)$	$\mathfrak{so}(8) \oplus \mathfrak{so}(2)$	$spin(7) \oplus \mathfrak{so}(2)$
15	$\mathfrak{so}(11)$	$\mathfrak{so}(8) \oplus \mathfrak{so}(3)$	$spin(7) \oplus \mathfrak{so}(3)$

TABLE 1. Riemannian g.o. spaces G/K fibered over irreducible symmetric spaces G/H ([63]).

5. GENERALIZED FLAG MANIFOLDS

In the work [3] D.V. Alekseevsky and the author classified generalized flag manifolds with homogeneous geodesics. Recall that a generalized flag manifold is a homogeneous space G/K which is an adjoint orbit of a compact semisimple Lie group G. Equivalently, the isotropy subgroup K is the centralizer of a torus (i.e. a maximal abelian subgroup) in G. We assume that G acts effectively on M. A flag manifold $M = G/K$ is simply connected and has the canonically defined decomposition $M = G/K = G_1/K_1 \times G_2/K_2 \times \cdots \times G_n/K_n$, where $G_1, \ldots, G_n$ are simple factors of the (connected) Lie group G. This decomposition is the de Rham decomposition of M equipped with a G-invariant metric g. In particular, (M, g) is a g.o. space if and only if each factor $(M_i = G_i/K_i, g_i = g|_{M_i})$ is a g.o. space. This reduces the problem of the description of G-invariant metrics with homogeneous geodesics in a flag manifold $M = G/K$ to the case when the group G is simple.

Flag manifolds $M = G/K$ with G a simple Lie group can be classified in terms of their *painted Dynkin diagrams*. It turns out

that for each classical Lie group there is an infinite series of flag manifolds, and for each of the exceptional Lie groups G_2, F_4, E_6, E_7, and E_8 there are 3, 11, 16, 31, and 40 non equivalent flag manifolds respectively (eg. [2], [16]). An important invariant of flag manifolds is their set of T-roots R_T. This is defined as the restriction of the root system R of $\mathfrak{g}$ to the center $\mathfrak{t}$ of the stability subalgebra $\mathfrak{k}$ of K. In [3] we defined the notion of *connected component* of R_T, namely two T-roots are in the same component if they can be connected by a chain of T-roots whose sum or difference is also a T-root. The set R_T is called *connected* if it has only one connected component.

Theorem 5.1. ([3]) *If the set of T-roots is connected then the standard metric on $M = G/K$ is the only G-invariant metric (up to scalar) which is a g.o. metric.*

Hence, for a flag manifold $M = G/K$ (G simple), a G-invariant g.o. metric may exist, only when R_T is not connected, so we only need to study those flag manifolds. It turns out that the system of T-roots is not connected only for three infinite series of a classical Lie group (namely the spaces $SO(2\ell + 1)/U(\ell - m) \cdot SO(2m + 1)$, $Sp(\ell)/U(\ell - m) \cdot Sp(m)$, and $SO(2\ell)/U(\ell - m) \cdot SO(2m)$), and for 10 flag manifolds of an exceptional Lie group. An perpective of the above theorem is given by the following theorem:

Theorem 5.2. ([3]) *Let $M = G/K$ be a flag manifold of a simple Lie group. Then the set of T-roots is not connected if and only if the isotropy representation of M consists of two irreducible (non-equivalent) components.*

Therefore, the problem of the description of G-invariant metrics on flag manifolds with homogeneous geodesics reduces substantially to the study of this short list of prospective flag manifolds. To this end, we used the classification Table 1 of the work of H. Tamaru ([63]). Since any flag manifold can be fibered over a symmetric space ([17]), then by using Theorem 5.2 we obtain that the only flag manifolds which are in Table 1 are $SO(2\ell+1)/U(\ell)$ and $Sp(\ell)/U(1) \cdot Sp(\ell-1)$.

On the other hand, in [5] D.N. Akhiezer and E.B. Vinberg classified all compact weakly symmetric spaces. Their classification shows that the only flag manifolds which are weakly symmetric spaces are $SO(2\ell + 1)/U(\ell)$ and $C(1, \ell - 1) = Sp(\ell)/U(1) \cdot Sp(\ell - 1)$. This implies that any $SO(2\ell+1)$-invariant metric g_λ on $SO(2\ell + 1)/U(\ell)$

(depending, up to scale, on one real parameter λ) is weakly symmetric, hence it has homogeneous geodesics. Similarly for any $\mathrm{Sp}(\ell)$-invariant metric g_λ on $\mathrm{Sp}(\ell)/U(1) \cdot \mathrm{Sp}(\ell-1)$. In fact, the action of the group $\mathrm{SO}(2\ell+1)$ on $\mathrm{SO}(2\ell+1)/\mathrm{U}(\ell)$ can be extended to the action of the group $\mathrm{SO}(2\ell+2)$ with isotropy subgroup $U(\ell+1)$, which preserves the complex structure and the standard invariant metric g_0 (which corresponds to $\lambda = 1$). Hence, the Riemannian flag manifold $(\mathrm{SO}(2\ell+1)/\mathrm{U}(\ell), g_0)$ is isometric to the Hermitian symmetric space $\mathrm{Com}(\mathbb{R}^{2\ell+2}) = \mathrm{SO}(2\ell+2)/\mathrm{U}(\ell+1)$ of all complex structures in $\mathbb{R}^{2\ell+2}$. Similarly, the action of the group $\mathrm{Sp}(\ell)$ on $\mathrm{Sp}(\ell)/U(1) \cdot \mathrm{Sp}(\ell-1)$ can be extended to the action of the group $\mathrm{SU}(2\ell)$ with isotropy subgroup $\mathrm{S}(U(1) \times U(2\ell-1))$. As a consequence of the above we obtain the following:

Theorem 5.3. ([3]) *The only flag manifolds $M = G/K$ of a simple Lie group G admitting a non naturally reductive G-invariant metric g with homogeneous geodesics are the manifolds $\mathrm{SO}(2\ell+1)/U(\ell)$ and $\mathrm{Sp}(\ell)/U(1) \cdot \mathrm{Sp}(\ell-1)$ $(\ell \geq 2)$, which admit (up to scale) a one-parameter family of $\mathrm{SO}(2\ell+1)$ (resp. $\mathrm{Sp}(\ell)$)-invariant metrics g_λ. Moreover, these manifolds are weakly symmetric spaces for $\lambda > 0$, and they are symmetric spaces with respect to $Isom(g_\lambda)$ if and only if $\lambda = 1$, i.e. g_λ is a multiple of the standard metric.*

Note that for $\ell = 2$ we obtain $\mathrm{Sp}(2)/U(1) \cdot \mathrm{Sp}(1) \cong \mathrm{SO}(5)/U(2)$, where the second quotient is an example of g.o. space in [45] which is not naturally reductive.

Finally, we mention a remarkable coincidence between Theorem 5.3 and a result by F. Podestà and G. Thorbergsson in [56], where they studied coisotropic actions on flag manifolds. One of their theorems states that if $M = G/K$ is a flag manifold of a simple Lie group then the action of K on M is coisotropic, if and only if M is up to local isomorphic either a Hermitian symmetric space, or one of the spaces obtained in Theorem 5.3.

6. GENERALIZED WALLACH SPACES

Let G/K be a compact homogeneous space with connected compact semisimple Lie group G and a compact subgroup K with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then G/K is called *generalized Wallach space* (known before as three-locally-symmetric spaces, cf.

[47]) if the module $\mathfrak{m}$ decomposes into a direct sum of three $\mathrm{Ad}(K)$-invariant irreducible modules pairwise orthogonal with respect to B, i.e. $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$, such that $[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{k}$ $i = 1, 2, 3$. Every generalized Wallach space admits a three parameter family of invariant Riemannian metrics determined by $\mathrm{Ad}(K)$-invariant inner products $\langle \cdot, \cdot \rangle = \lambda_1 B(\cdot, \cdot) \mid_{\mathfrak{m}_1} + \lambda_2 B(\cdot, \cdot) \mid_{\mathfrak{m}_2} + \lambda_3 B(\cdot, \cdot) \mid_{\mathfrak{m}_3}$, where $\lambda_1, \lambda_2, \lambda_3$ are positive real numbers. The classification of generalized Wallach spaces was recently obtained by Yu.G. Nikoronov ([52]) (G semisimple) and Z. Chen, Y. Kang, K. Liang ([18]) (G simple) as follows:

Theorem 6.1 ([52]). *Let G/K be a connected and simply connected compact homogeneous space. Then G/K is a generalized Wallach space if and only if it is one of the following types:*

1) G/K is a direct product of three irreducible symmetric spaces of compact type.

2) The group is simple and the pair $(\mathfrak{g}, \mathfrak{k})$ is one of the pairs in Table 2.

3) $G = F \times F \times F \times F$ and $K = diag(F) \subset G$ for some connected, compact, simple Lie group F, with the following description on the Lie algebra level:

$$(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{f} \oplus \mathfrak{f} \oplus \mathfrak{f} \oplus \mathfrak{f}, \mathrm{diag}(\mathfrak{f})) = \{(X, X, X, X) \mid X \in f\},$$

where $\mathfrak{f}$ is the Lie algebra of F, and (up to permutation) $\mathfrak{m}_1 = \{(X, X, -X, -X) \mid X \in f\}$, $\mathfrak{m}_2 = \{(X, -X, X, -X) \mid X \in f\}$, $\mathfrak{m}_3 = \{(X, -X, -X, X) \mid X \in f\}$.

$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{g}$	$\mathfrak{k}$
$\mathfrak{so}(k+l+m)$	$\mathfrak{so}(k) \oplus \mathfrak{so}(l) \oplus \mathfrak{so}(m)$	$\mathfrak{e}_7$	$\mathfrak{so}(8) \oplus 3\mathfrak{sp}(1)$
$\mathfrak{su}(k+l+m)$	$\mathfrak{su}(k) \oplus su(l) \oplus \mathfrak{su}(m)$	$\mathfrak{e}_7$	$\mathfrak{su}(6) \oplus \mathfrak{sp}(1) \oplus \mathbb{R}$
$\mathfrak{sp}(k+l+m)$	$\mathfrak{sp}(k) \oplus \mathfrak{sp}(l) \oplus \mathfrak{sp}(m)$	$\mathfrak{e}_7$	$\mathfrak{so}(8)$
$\mathfrak{su}(2l), l \geq 2$	$\mathfrak{u}(l)$	$\mathfrak{e}_8$	$\mathfrak{so}(12) \oplus 2\mathfrak{sp}(1)$
$\mathfrak{so}(2l), l \geq 4$	$\mathfrak{u}(l) \oplus \mathfrak{u}(l-1)$	$\mathfrak{e}_8$	$\mathfrak{so}(8) \oplus \mathfrak{so}(8)$
$\mathfrak{e}_6$	$\mathfrak{su}(4) \oplus 2\mathfrak{sp}(1) \oplus \mathbb{R}$	$\mathfrak{f}_4$	$\mathfrak{so}(5) \oplus 2\mathfrak{sp}(1)$
$\mathfrak{e}_6$	$\mathfrak{so}(8) \oplus \mathbb{R}^2$	$\mathfrak{f}_4$	$\mathfrak{so}(8)$
$\mathfrak{e}_6$	$\mathfrak{sp}(3) \oplus \mathfrak{sp}(1)$		

Table 2. The pairs $(\mathfrak{g}, \mathfrak{k})$ corresponding to generalized Wallach spaces G/K with G simple ([52]).

In [10] Yu Wang and the author investigated which of the families of spaces listed in Theorem 6.1 are g.o. spaces. By applying the method of searching for geodesics vectors shown at the end of Section 2 we obtained the following:

Theorem 6.2. ([10]) *Let* $(G/K, g)$ *be a generalized Wallach space as listed in Theorem 6.1. Then*

1) If $(G/K, g)$ *is a space of type 1) then this is a g.o. space for any* $\mathrm{Ad}(K)$*-invariant Riemannian metric.*

2) If $(G/K, g)$ *is a space of type 2) or 3) then this is a g.o. space if and only if* g *is the standard metric.*

However, in order to find all homogeneous geodesics in G/K it suffices to find all the real solutions of a system of $\dim \mathfrak{m}_1 + \dim \mathfrak{m}_2 + \dim \mathfrak{m}_3$ quadratic equations. By Theorem 6.2 we only need to consider homogeneous geodesics for spaces of types 2) and 3) given in Theorem 6.1, for the metric $(\lambda_1, \lambda_2, \lambda_3)$, where at least two of $\lambda_1, \lambda_2, \lambda_3$ are different. This is not easy in general. We obtained all homogeneous geodesics (for various values of the parameters $\lambda_1, \lambda_2, \lambda_3$ for the generalized Wallach space $SU(2)/\{e\}$, hence recovering a result on R.A. Marinosci ([49, p. 266]), and for the Stiefel manifolds $\mathrm{SO}(n)/\mathrm{SO}(n-2)$, $(n \geq 4)$.

7. M-SPACES

Let G/K be a generalized flag manifold with $K = C(S) = S \times K_1$, where S is a torus in a compact simple Lie group G and K_1 is the semisimple part of K. Then the *associated M-space* is the homogeneous space G/K_1. These spaces were introduced and studied by H.C. Wang in [64].

In the works [11] and [12]Y. Wang, G. Zhao and the author investigated homogeneous geodesics in a class of homogeneous spaces called M-spaces. We proved that for various classes of M-spaces, the only g.o. metric is the standard metric. For other classes of M-spaces we give either necessary or necessary and sufficient conditions so that a G-invariant metric on G/K_1 is a g.o. metric. The analysis is based on properties of the isotropy representation $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$ of the flag manifold G/K, in particular on the dimension of the submodules $\mathfrak{m}_i$. We summarize these results below.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of the Lie groups G and K respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be an $\mathrm{Ad}(K)$-invariant reductive decomposition

of the Lie algebra $\mathfrak{g}$, where $\mathfrak{m} \cong T_o(G/K)$. This is orthogonal with respect to $B = -$Killing from on $\mathfrak{g}$. Assume that

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s \qquad (6)$$

is a B-orthogonal decomposition of $\mathfrak{m}$ into pairwise inequivalent irreducible ad($\mathfrak{k}$)-modules.

Let G/K_1 be the corresponding M-space and $\mathfrak{s}$ and $\mathfrak{k}_1$ be the Lie algebras of S and K_1 respectively. We denote by $\mathfrak{n}$ the tangent space $T_o(G/K_1)$, where $o = eK_1$. A G-invariant metric g on G/K_1 induces a scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$ which is $\mathrm{Ad}(K_1)$-invariant. Such an $\mathrm{Ad}(K_1)$-invariant scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$ can be expressed as $\langle x, y \rangle = B(\Lambda x, y)$ $(x, y \in \mathfrak{n})$, where Λ is the $\mathrm{Ad}(K_1)$-equivariant positive definite symmetric operator on $\mathfrak{n}$.

The main results are the following:

Theorem 7.1. ([11]) *Let G/K be a generalized flag manifold with $s \geq 3$ in the decomposition (6). Let G/K_1 be the corresponding M-space.*

1) If $\dim \mathfrak{m}_i \neq 2$ $(i = 1, \ldots, s)$ and $(G/K_1, g)$ is a g.o. space, then

$$g = \langle \cdot, \cdot \rangle = \mu B(\cdot, \cdot) \mid_{\mathfrak{s}} + \lambda B(\cdot, \cdot) \mid_{\mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \cdots \oplus \mathfrak{m}_s}, \ (\mu, \lambda > 0).$$

2) If there exists some $j \in \{1, \ldots, s\}$ such that $\dim \mathfrak{m}_j = 2$, then $(G/K_1, g)$ is a g.o. space if and only if g is the standard metric.

Theorem 7.2. ([12]) *Let G/K be a generalized flag manifold with two isotropy summands given by (6), and $(G/K_1, g)$ be the corresponding M-space. Then*

1) If $\dim \mathfrak{m}_2 = 2$, then the standard metric is the only g.o. metric on M-space $(G/K_1, g)$, unless $G/K_1 = SO(5)/SU(2)$ or $Sp(n)/Sp(n-1)$, $(n \geq 2)$.

2) If $\dim \mathfrak{m}_2 \neq 2$ and the M-space $(G/K_1, g)$ is a g.o. space, then $g = \langle \cdot, \cdot \rangle = \mu B(\cdot, \cdot) \mid_{\mathfrak{s}} + \lambda B(\cdot, \cdot) \mid_{\mathfrak{m}_1 \oplus \mathfrak{m}_2}, \ (\mu, \lambda > 0)$, unless $G/K_1 = SO(2n+1)/SU(n), (n > 2)$.

However, the spaces $SO(5)/SU(2)$ and $Sp(n)/Sp(n-1)$ are included in Tamaru's Table 1, therefore they admit g.o. metrics. For the generalized flag manifolds with $s = 1$ or 2 in the decomposition (6) we use Theorem 6.2 and Tamaru's results in [63] to prove existence of non naturally reductive g.o. metrics in certain M-spaces, including the three isolated classes listed in parts 1) and 2) of Theorem 6.2.

We prove the following:

Theorem 7.3. ([12]) *The M-spaces $SU(n+1)/SU(n)$, $(n \geq 2)$, $SU(2r+n)/SU(r) \times SU(r+n)$, $(r \geq 2, n \geq 1)$, $SO(4n+1)/SU(2n)$, $(n \geq 1)$, $Sp(n)/Sp(n-1), (n \geq 2)$, $SO(4n+2)/SU(2n+1)$, $(n \geq 2)$ and $E_6/SO(8)$ admit non naturally reductive g.o. metrics.*

Finally, by using techniques from [11] we can prove the following:

Theorem 7.4. ([11]) *Let G/K be a generalized flag manifold with corresponding M-space $(G/K_1, g)$.*
1) If $G = G_2$, then $(G_2/K_1, g)$ is a g.o. space if and only if g is the standard metric.
2) If $G = F_4$, then the standard metric is the only g.o. metric on F_4/K_1, unless $K_1 = SU(2) \times SU(3)$, or $K_1 = SO(7)$.
3) If $G = E_6$, then the standard metric is the only g.o. metric on E_6/K_1, unless K_1 is one of $SU(3) \times SU(3) \times SU(2)$, $SU(5) \times SU(2)$, $SU(2) \times SU(2) \times SU(3)$, $SO(8)$, or $SO(10)$.

By a result of H. Tamaru [63] it follows that the M-space $E_6/SO(10)$ admits non-naturally reductive g.o. metrics.

8. HOMOGENEOUS GEODESICS IN PSEUDO-RIEMANNIAN MANIFOLDS

It is well known that any homogeneous Riemannian manifold is reductive, but this is not the case for pseudo-Riemannian manifolds in general. In fact, there exist homogeneous pseudo-Riemannian manifolds which do not admit any reductive decomposition. Therefore, there is a dichotomy in the study of geometrical problems between reductive and non reductive pseudo-Riemannian manifolds. Due to the existence of null vectors in a pseudo-Riemannian manifold the definition of a homogeneous geodesic $\gamma(t) = \exp(tX) \cdot o$ needs to be modified by requiring that $\nabla_{\dot{\gamma}} \dot{\gamma} = k(\gamma) \dot{\gamma}$ (see also relevant discussion in [48, pp. 90-91]). It turns out that $k(\gamma)$ is a constant function (cf. [31]. Even though an algebraic characterization of geodesic vectors (that is an analogue of the geodesic Lemma 2.1) was known to physicists ([35], [55]), a formal proof was given by Z. Dušek and O. Kowalski in [31].

Lemma 8.1 ([31]). *Let $M = G/H$ be a reductive homogeneous pseudo-Riemannian space with reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$,*

and $X \in \mathfrak{g}$. Then the curve $\gamma(t) = \exp(tX) \cdot o$ is a geodesic curve with respect to some parameter s if and only if

$$\langle [X, Z]_\mathfrak{m}, X_\mathfrak{m} \rangle = k \langle X_\mathfrak{m}, Z_\mathfrak{m} \rangle, \quad for \ all \ \ Z \in \mathfrak{m},$$

where k is some real constant. Moreover, if $k = 0$, then t is an affine parameter for this geodesic. If $k \neq 0$, then $s = e^{kt}$ is an affine parameter for the geodesic. This occurs only if the curve $\gamma(t)$ is a null curve in a (properly) pseudo-Riemannian space.

For applications of this lemma see [28]. The existence of homogeneous geodesics in homogeneous pseudo-Riemannian spaces (for both reductive and non reductive) was answered positively only recently by Z. Dušek in [27].

Two-dimensional and three-dimensional homogeneous pseudo Riemannian manifolds are reductive ([19], [34]). Four-dimensional non reductive homogeneous pseudo-Riemannian manifolds were classified by M.E. Fels and A.G. Renner in [34] in terms of their non reductive Lie algebras. Their invariant pseudo-Riemannian metrics, together with their connection and curvature, were explicitly described in by G. Calvaruso and A. Fino in [22].

The three-dimensional pseudo-Riemannian g.o. spaces were classified by G. Calvaruso and Marinosci in [21]. In the recent work [23], G. Calvaruso, A. Fino and A. Zaeim obtained explicit examples of four-dimensional non reductive pseudo-Riemannian g.o. spaces. They deduced an explicit description in coordinates for all invariant metrics of non reductive homogeneous pseudo-Riemannian four-manifolds. For those four-dimensional non reductive pseudo-Riemannia spaces which are not g.o., they determined the homogeneous geodesics though a point.

9. Two-step homogeneous geodesics

In the work [9] N.P. Souris and the author considered a generalisation of homogeneous geodesics, namely geodesics of the form

$$\gamma(t) = \exp(tX) \exp(tY) \cdot o, \quad X, Y \in \mathfrak{g}, \tag{7}$$

which we named *two-step homogeneous geodesics*. We obtained sufficient conditions on a Riemannian homogeneous space G/K, which imply the existence of two-step homogeneous geodesics in G/K. A Riemannian homogeneous spaces G/K such that any geodesic of

G/K passing through the origin is two-step homogeneous is called *two-step g.o. spaces.*

Geodesics of the form (7) had appeared in the work [65] of H.C. Wang as geodesics in a semisimple Lie group G, equipped with a metric induced by a Cartan involution of the Lie algebra $\mathfrak{g}$ of G. Also, in [25] R. Dohira proved that if the tangent space $T_o(G/K)$ of a homogeneous space splits into submodules $\mathfrak{m}_1, \mathfrak{m}_2$ satisfying certain algebraic relations, and if G/K is endowed with a special one parameter family of Riemannian metrics g_c, then all geodesics of the Riemannian space $(G/K, g_c)$ are of the form (7). The main result of [9] is the following:

Theorem 9.1. *([9]) Let $M = G/K$ be a homogeneous space admitting a naturally reductive Riemannian metric. Let B be the corresponding inner product on $\mathfrak{m} = T_o(G/K)$. We assume that $\mathfrak{m}$ admits an $\mathrm{Ad}(K)$-invariant orthogonal decomposition*

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \cdots \oplus \mathfrak{m}_s, \qquad (8)$$

with respect to B. We equip G/K with a G-invariant Riemannian metric g corresponding to the $\mathrm{Ad}(K)$-invariant positive definite inner product $\langle \cdot, \cdot \rangle = \lambda_1 B|_{\mathfrak{m}_1} + \cdots + \lambda_s B|_{\mathfrak{m}_s}$, $\lambda_1, \ldots, \lambda_s > 0$. If $(\mathfrak{m}_a, \mathfrak{m}_b)$ is a pair of submodules in the decomposition (8) such that

$$[\mathfrak{m}_a, \mathfrak{m}_b] \subset \mathfrak{m}_a, \qquad (9)$$

then any geodesic γ of $(G/K, g)$ with $\gamma(0) = o$ and $\dot{\gamma}(0) \in \mathfrak{m}_a \oplus \mathfrak{m}_b$, is a two-step homogeneous geodesic. In particular, if $\dot{\gamma}(0) = X_a + X_b \in \mathfrak{m}_a \oplus \mathfrak{m}_b$, then for every $t \in \mathbb{R}$ this geodesic is given by

$$\gamma(t) = \exp t(X_a + \lambda X_b) \exp t(1 - \lambda) X_b \cdot o, \quad \text{where } \lambda = \lambda_b / \lambda_a.$$

Moreover, if either $\lambda_a = \lambda_b$ or $[\mathfrak{m}_a, \mathfrak{m}_b] = \{0\}$ holds, then γ is a homogeneous geodesic, that is $\gamma(t) = \exp t(X_a + X_b) \cdot o$, for any $t \in \mathbb{R}$.

The following corollary provides a method to obtain many examples of two-step g.o. spaces.

Corollary 9.2. *Let $M = G/K$ be a homogeneous space admitting a naturally reductive Riemannian metric. Let B be the corresponding inner product of $\mathfrak{m} = T_o(G/K)$. We assume that $\mathfrak{m}$ admits an $\mathrm{Ad}(K)$-invariant, B-orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, such*

that $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1$. *Then M admits an one-parameter family of G-invariant Riemannian metrics g_λ, $\lambda \in \mathbb{R}^+$, such that (M, g_λ) is a two-step g.o. space. Each metric g_λ corresponds to an* $\mathrm{Ad}(K)$-*invariant positive definite inner product on $\mathfrak{m}$ of the form* $\langle\ ,\ \rangle = B|_{\mathfrak{m}_1} + \lambda\, B|_{\mathfrak{m}_2}$, $\lambda > 0$.

The above Corollary 9.2 is a generalisation of Dohira's result [25].

The main tool for the proof of Theorem 9.1 is the following proposition.

Proposition 9.3. ([8])*Let $M = G/K$ be a homogeneous space and $\gamma : \mathbb{R} \to M$ be the curve $\gamma(t) = \exp(tX)\exp(tY)\exp(tZ) \cdot o$, where $X, Y, Z \in \mathfrak{m}$. Let $T : \mathbb{R} \to \mathrm{Aut}(\mathfrak{g})$ be the map given by $T(t) = \mathrm{Ad}(\exp(-tZ)\exp(-tY))$. Then γ is a geodesic in M through $o = eK$ if and only if for any $W \in \mathfrak{m}$, the function $G_W : \mathbb{R} \to \mathbb{R}$ given by*

$$
\begin{aligned}
G_W(t) \quad &= \quad \langle\, (TX)_\mathfrak{m} + (TY)_\mathfrak{m} + Z_\mathfrak{m}, [W, TX + TY + Z]_\mathfrak{m} \,\rangle \\
&\quad + \ \langle\, W, [TX, TY + Z]_\mathfrak{m} + [TY, Z]_\mathfrak{m} \,\rangle,
\end{aligned}
$$

is identically zero, for every $t \in \mathbb{R}$.

The above proposition is a new tool towards the study of geodesics consisting of more than one exponential factors. In fact, for $X = Y = 0$ we obtain Lemma 2.1 of Kowalski and Vanhecke.

A natural application of Corollary 9.2 is for total spaces of homogeneous Riemannian submersions, as shown below.

Proposition 9.4. *Let G be a Lie group admitting a bi-invariant Riemannian metric and let K, H be closed and connected subgroups of G, such that $K \subset H \subset G$. Let B be the Ad-invariant positive definite inner product on the Lie algebra $\mathfrak{g}$ corresponding to the bi-invariant metric of G. We identify each of the spaces $T_o(G/K)$, $T_o(G/H)$ and $T_o(H/K)$ with corresponding subspaces $\mathfrak{m}, \mathfrak{m}_1$ and $\mathfrak{m}_2$ of $\mathfrak{g}$, such that $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. We endow G/K with the G-invariant Riemannian metric g_λ corresponding to the $\mathrm{Ad}(K)$-invariant positive definite inner product $\langle\ ,\ \rangle = B|_{\mathfrak{m}_1} + \lambda\, B|_{\mathfrak{m}_2}$, $\lambda > 0$. Then $(G/K, g_\lambda)$ is a two-step g.o. space.*

Example 9.5. ([8]) The odd dimensional sphere $\mathbb{S}^{2n+1}$ can be considered as the total space of the homogeneous Hopf bundle $\mathbb{S}^1 \to \mathbb{S}^{2n+1} \to \mathbb{C}P^n$. Let g_1 be the standard metric of $\mathbb{S}^{2n+1}$. We equip $\mathbb{S}^{2n+1}$ with an one parameter family of metrics g_λ, which "deform"

the standard metric along the Hopf circles $\mathbb{S}^1$. By setting $G = U(n+1)$, $K = U(n)$ and $H = U(n) \times U(1)$, the Hopf bundle corresponds to the fibration $H/K \to G/K \to G/H$.

Since $U(n+1)$ is compact, it admits a bi-invariant metric corresponding to an $\mathrm{Ad}(U(n+1))$-invariant positive definite inner product B on $\mathfrak{u}(n+1)$. We identify each of the spaces $T_o\mathbb{S}^{2n+1} = T_o(G/K), T_o\mathbb{C}P^n = T_o(G/H)$, and $T_o\mathbb{S}^1 = T_o(H/K)$ with corresponding subspaces $\mathfrak{m}, \mathfrak{m}_1$, and $\mathfrak{m}_2$ of $\mathfrak{u}(n+1)$. The desired one parameter family of metrics g_λ corresponds to the one parameter family of positive definite inner products $\langle \; , \; \rangle = B|_{\mathfrak{m}_1} + \lambda\, B|_{\mathfrak{m}_2}$, $\lambda > 0$ on $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Then Proposition 9.4 implies that $(\mathbb{S}^{2n+1}, g_\lambda)$ is a two-step g.o. space. In particular, let $X \in T_o\mathbb{S}^{2n+1}$. Then the unique geodesic γ of $(\mathbb{S}^{2n+1}, g_\lambda)$ with $\gamma(0) = o$ and $\dot\gamma(0) = X$, is given by $\gamma(t) = \exp t(X_1 + \lambda X_2) \exp t(1 - \lambda) X_2 \cdot o$, where X_1, X_2 are the projections of X on $\mathfrak{m}_1 = T_o\mathbb{C}P^n$ and $\mathfrak{m}_2 = T_o\mathbb{S}^1$ respectively. Note that if $\lambda = 1 + \epsilon$, $\epsilon > 0$, then the metrics $g_{1+\epsilon}$ are Cheeger deformations of the natural metric g_1.

By using Proposition 9.2 it is possible to construct various classes of two-step g.o spaces. The recipe is the following:
(i) Let G/K be a homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ admitting a naturally reductive metric corresponding to a positive definite inner product B on $\mathfrak{m}$.
(ii) We consider an $\mathrm{Ad}(K)$-invariant, orthogonal decomposition $\mathfrak{m} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_s$ with respect to B.
(iii) We separate the submodules $\mathfrak{n}_i$ into two groups as $\mathfrak{m}_1 = \mathfrak{n}_{i_1} \oplus \cdots \oplus \mathfrak{n}_{i_n}$ and $\mathfrak{m}_2 = \mathfrak{n}_{i_{n+1}} \oplus \cdots \oplus \mathfrak{n}_{i_s}$, so that $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1$. The decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ is $\mathrm{Ad}(K)$-invariant and orthogonal with respect to B.
(iv) Then Corollary 9.2 implies that G/K admits an one parameter family of metrics g_λ so that $(G/K, g_\lambda)$ is a two-step g.o. space.

In [9] we applied the above recipe to the following classes of homogeneous spaces:
1)   Lie groups with bi-invariant metrics, equipped with an one parameter family of left-invariant metrics.
2)   Flag manifolds equipped with certain one parameter families of diagonal metrics.
3)   Generalized Wallach spaces equipped with three different types of diagonal metrics (thus recovering some results from [8]).

4) k-symmetric spaces where k is even, endowed with an one parameter family of diagonal metrics.

10. Some open problems

It seems that the target for a complete classification of homogeneous g.o. spaces in any dimension greater than seven is far for being accomplished. In dimension seven there are several examples but a complete classification is still unknown. However, as shown in the present paper, for some large classes of homogeneous spaces it is possible to obtain some necessary conditions for the g.o. property. These conditions are normally imposed by the special Lie theoretic structure of corresponding homogeneous space. Also, the problem of an explicit description of homogeneous geodesics for spaces which are not g.o., is not trivial either. Eventhough it is mathematically simple, it requires high computational complexity. A more tractable problem could be to classify g.o. spaces with two or three irreducible isotropy summands.

Further, it is not usually an easy matter to show that the g.o. property of $(M = G/K, g)$ does not depend on the representation as a coset space and on the $\mathrm{Ad}(K)$-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Therefore, we often stress that we study G-g.o. spaces.

Also, it would be interesting to see how various results about Riemannian manifolds could be adjusted to pseudo-Riemannian manifolds, such as Propositions 2.2, 9.3.

Concerning generalizations of the g.o. property, we have introduced the concept of a two-step homogeneous geodesic and two-step g.o. space. We conjecture that a search for three-step (or more) homogeneous geodesics reduces to two-step homogeneous geodesics. Also, it would be interesting to study two-step homogeneous geodesics in pseudo-Riemannian manifolds (formulate corresponding geodesic lemma etc.).

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