Irish Math. Soc. Bulletin
Number 67, Summer 2012, 67–70
ISSN 0791-5578

PROBLEMS

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The first two problems were contributed by Finbarr Holland.

Problem 69.1. Suppose that the matrices A, b, and c are of sizes $n \times n$, $n \times 1$, and $1 \times n$, respectively. Prove that, for all complex numbers z,

$$\det(A - zbc) = \det A - zcA^*b = \det A + z\det \begin{pmatrix} 0 & c \\ b & A \end{pmatrix},$$

where A^* is the adjoint of A (that is, the transpose of the matrix of cofactors of A).

Problem 69.2. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \sum_{k=1}^{n} \frac{1}{k} = \zeta(3),$$

where ζ is the Riemann zeta function.

I came across the final problem as a graduate student.

Problem 69.3. A rectangle is partitioned into finitely many smaller rectangles. Each of these smaller rectangles has a side of integral length. Prove that the larger rectangle also has a side of integral length.

Here are the solutions to the problems from *Bulletin* Number 67. The first solution was contributed by the North Kildare Mathematics Problem Club.

Problem 67.1. Prove that there does not exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$ that satisfies

$$f'(x) \geqslant 1 + [f(x)]^2$$

for each real number x.

Received on 24-8-2012.

Solution 67.1. Suppose there were such an f. Let

$$g(x) = \arctan f(x) - x.$$

Then

$$g'(x) = \frac{f'(x)}{1 + f(x)^2} - 1 \geqslant 0$$

for all $x \in \mathbb{R}$, so g is nondecreasing, so for $x > 0$ we have

$$\arctan f(x) \geqslant x + \arctan f(0).$$

This is impossible, because $\arctan f(x) < \pi/2$ for all real numbers x. Therefore no such function f exists. □

The second solution was shown to me some years ago by Edward Crane, shortly after he was given the problem.

*Problem 67.2***.** Suppose that $x_1, x_2, \ldots, x_n$, where $n \geqslant 3$, are non-negative real numbers such that

$$x_1 + x_2 + \cdots + x_n = 2$$

and

$$x_1x_2 + x_2x_3 + \cdots + x_{n-2}x_{n-1} + x_{n-1}x_n = 1.$$

Find the maximum and minimum values of

$$x_1^2 + x_2^2 + \cdots + x_n^2.$$

Solution 67.2. Let

$$A = \sum_{\substack{i \leqslant n \\ i \text{ odd}}} x_i \quad \text{and} \quad B = \sum_{\substack{i \leqslant n \\ i \text{ even}}} x_i.$$

Then $A + B = 2$, so $AB \leqslant 1$, and hence

$$1 = x_1x_2 + x_2x_3 + \cdots + x_{n-2}x_{n-1} + x_{n-1}x_n \leqslant AB \leqslant 1.$$

Equality in the first inequality implies that all terms x_j are 0 other than three consecutive terms x_{i-1}, x_i, and x_{i+1} (and one of these may be 0). This reduces the problem to the $n = 3$ case. In this case you can easily check that $x_2 = 1$, and the minimum is 3/2 and the maximum is 2. □

The third solution was contributed by the North Kildare Mathematics Problem Club (they also submitted an alternative solution to the second problem).

*Problem 67.3***.** There are m gold coins divided unequally between n chests. An enormous queue of people are asked in turn to select a chest. Each member of the queue knows how many coins there are in each chest, and also knows the choice of those ahead in the queue who have selected already. In choosing a chest, each person considers the (possibly non-integer) number of gold coins he would receive were the coins in that chest to be shared equally amongst all those, including him, who have selected that chest so far. He then chooses the chest that maximises this number of coins. For example, if there are three chests A, B, and C containing 3, 5, and 8 coins, then the first person in the queue selects C, the second selects B, the third selects C, the fourth selects A, and so forth.

After the mth person has chosen a chest, how many people have selected each chest? Express your answer in terms of the number of coins per chest. What more can be said about people's chest selections?

Solution 67.3. We claim that the number of people who choose each chest by the m-th stage is equal to the number of coins in the chest.

We remark that the choice of chest is not always uniquely determined. However, this does not affect the state of play after m choices.

Suppose c_j coins are in chest j. Let $p_j^0 = 0$ for all j. For $n \geq 1$, let p_j^n be the number who have chosen chest j when the n-th person has made his choice.

Our claim is that $p_j^m = c_j$ for each j.

Suppose some $p_k^m > c_k$. Let n be the first number with $p_k^n > c_k$. Then

$$\sum_{j \neq k} p_j^n = n - p_k^n < m - c_k = \sum_{j \neq k} c_j,$$

so there exists some j with $p_j^n < c_j$. But then chest j would have been a better choice than chest k at the n-th stage, since

$$\frac{c_j}{p_j^n + 1} \geqslant 1 > \frac{c_k}{p_k^n}.$$

So this is impossible.

Thus $p_j^m \leqslant c_j$ for each j, and since

$$m = \sum_j p_j^m \leqslant \sum_j c_j = m,$$

we conclude that $p_j^m = c_j$ for each j, as claimed.

More generally, regarding each coin as a packet of r coins, we see that when $n = mr$, the number choosing chest j is rc_j. $\square$

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com`.

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