Irish Mathematical Society
Cumann Matamaitice na hÉireann

Bulletin

Number 69 **Summer 2012**

ISSN 0791-5578

Irish Mathematical Society Bulletin

Editor: Anthony G. O'Farrell

The aim of the *Bulletin* is to inform Society members, and the mathematical community at large, about the activities of the Society and about items of general mathematical interest. It appears twice each year. The *Bulletin* is supplied free of charge to members; it is sent abroad by surface mail. Libraries may subscribe to the *Bulletin* for 30 euro per annum.

The *Bulletin* seeks articles written in an expository style and likely to be of interest to the members of the Society and the wider mathematical community. We encourage informative surveys, biographical and historical articles, short research articles, classroom notes, Irish thesis abstracts, book reviews and letters. All areas of mathematics will be considered, pure and applied, old and new. See the inside back cover for submission instructions.

Correspondence concerning the *Bulletin* should be sent, in the first instance, by e-mail to the Editor at

`ims.bulletin@gmail.com`

and only if not possible in electronic form to the address

The Editor
Irish Mathematical Society Bulletin
Department of Mathematics and Statistics
NUI, Maynooth
Co. Kildare

Further information about the Irish Mathematical Society and its *Bulletin* can be obtained from the IMS webpage

`http://www.maths.tcd.ie/pub/ims/`

Printed in the University of Limerick

EDITORIAL

It is a great sadness to me and to many in the Irish mathematical community that we have lost Jim Flavin, who for as long as I can remember was a constant support to younger mathematicians and who embodied the highest standards in Applied Mathematics. His work on nonlinear differential equations was world-class. MathSciNet lists 57 papers[1]. His love and support for the Irish language were also well-known, and as a member of the scientific committee of An Coiste Téarmaíochta he invested a great deal of time and attention in publications such as 'Foclóir Eolaíochta' (Eagrán Méadaithe, An Gúm 1994) and 'Téarmaí Ríomhaireachta' (An Gúm 1990), which are vital supports for those who wish to conduct scientific discourse in the language. He was working lately on an expanded biography of Pádraig De Brún, that colourful and passionate polymath, prankster, poet and visionary, the only full professor of mathematics so far arrested for plotting the overthrow of the Irish state, who served in Maynooth, DIAS and finally as President of UCG. Jim's work on the biographical note published on the occasion of the 50th anniversay of DIAS [2] gathered much valuable material, and it is to hoped that whatever else he put together more recently can be brought to light.

Matt McCarthy has compiled an obituary note that appears in this issue. *Ní fheicfimid a leithéid arís.*

Following considerable distress and controversy in the mathematical and wider scientific community about a policy shift in Science Foundation Ireland which resulted in the "administrative" rejection of many grant proposals (i.e. rejection before consideration by any scientific peers), the National Committee for Mathematical Science decided to articulate these concerns. The Chair, Richard Timoney, wrote to the SFI chief, Mark Ferguson. A copy of the letter

[1] See `http:http://www.ams.org/mathscinet/search/publications.html?pg1=INDI&s1=67460` (if you have access).

[2] Irishleabhar Mhá Nuad 1994, pp. 9-32; It was also published in DIAS School of Theoretical Physics 50 Year Report, pp. 10-29 with two photographs and a facsimile of the title-page of De Brún's thesis

may be viewed on the Royal Irish Academy website[3]. Earlier, the RIA President, Luke Drury, spoke publicly about the importance of supporting fundamental research in basic science, including mathematics. Dr. Ferguson has agreed to meet Professor Timoney and a delegation, and it is hoped that the situation will improve.

The Bulletin is exchanged for the publications of a good few learned societies. This has the benefit of widening its impact, and the exchange material that comes in is distributed among Irish institutions. In some cases, we are more motivated by the desire to assist poorly-resourced colleagues overseas. Where the incoming journals go is determined by the IMS Committee, taking into account expressed preferences. We have recently agreed a new exchange with for the Iranian Journal of Mathematical Chemistry. We have a run of Note di Matematica (published by Universita del Salento) and this has not yet been allocated. Expressions of interest in either periodical are invited. We have long had an exchange with the Deutsche Mathematiker Vereinigung. Our Vice-President Martin Mathieu reports that the agreement with the DMV has been extended to a reciprocity agreement that allows members of the IMS to become members of the DMV at a reduced rate (and vice versa).

Tony Wickstead points out errata in the Winter 2011 number: On page 15, the report of Duncan Lawson's talk is cut short (actually in the middle of a line!), and at the foot of page 16, the line 'Martin Stynes, University College Cork' should really have been moved to the top of page 17. I regret these lapses, and will endeavour to do better. If I fail, corrections are always welcome.

In the last issue, I invited schools to send links to contact points for prospective research students in Mathematics. These are the ones that have come in so far:

DCU: Olaf Menkens `http://www.dcu.ie/info/staff_member.php?id_no=2659`
NUIG: Jim Cruickshank `mailto://jam:es.cruickshank@nuigalway.ie`
NUIM: `http://www.maths.nuim.ie/pghowtoapply`
QUB: `http://www.qub.ac.uk/puremaths/Funded_PG_2012.html`

[3] `http://ria.ie/getmedia/cb045886-692e-4d2c-baba-3760c9507368/Letter-to-SFI.pdf.aspx`

TCD: http://www.maths.tcd.ie/postgraduate/
UCD: Nuria Garcia Ordiales mailto://nuria.garcia@ucd.ie
UU: http://www.compeng.ulster.ac.uk/rgs/

I again invite the remaining schools with Ph.D. programmes in Mathematics to send me their preferred link, a url that works. I remind readers of the print edition that all links are live, and hence may be accessed by a click, in the electronic edition of this Bulletin[4].

AOF. Department of Mathematics and Statistics, NUI, Maynooth, Co. Kildare

E-mail address: ims.bulletin@gmail.com

[4] http://www.maths.tcd.ie/pub/ims/bulletin/

NOTICES FROM THE SOCIETY

Officers and Committee Members

President	Dr S. Wills	Dept. of Mathematics NUI Cork
Vice-President	Dr M. Mathieu	Dept. of Pure Mathematics QUB
Secretary	Dr R. Quinlan	Dept. of Mathematics NUI Galway
Treasurer	Dr S. Breen	Dept. of Mathematics St Patrick's College Drumcondra

Prof S. Buckley, Dr J. Cruickshank, Dr. J. Gleeson, Dr C. Hills, Dr. Bernd Kreussler, Dr M. Mackey, Dr S. O'Rourke, Dr C. Stack, Prof A. Wickstead.

Local Representatives

Belfast	QUB	Dr M. Mathieu
Carlow	IT	Dr D. Ó Sé
Cork	IT	Dr D. Flannery
	UCC	Prof. M. Stynes
Dublin	DIAS	Prof Tony Dorlas
	DIT	Dr C. Hills
	DCU	Dr M. Clancy
	St Patrick's	Dr S. Breen
	TCD	Prof R. Timoney
	UCD	Dr R. Higgs
Dundalk	IT	Mr Seamus Bellew
Galway	UCG	Dr J. Cruickshank
Limerick	MIC	Dr B. Kreussler
	UL	Mr G. Lessells
Maynooth	NUI	Prof S. Buckley
Tallaght	IT	Dr C. Stack
Tralee	IT	Dr B. Guilfoyle
Waterford	IT	Dr P. Kirwan

Applying for I.M.S. Membership

(1) The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Deutsche Mathematiker Vereinigung, the Irish Mathematics Teachers Association, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

(2) The current subscription fees are given below:

Institutional member	160 euro
Ordinary member	25 euro
Student member	12.50 euro
DMV, I.M.T.A., NZMS or RSME reciprocity member	12.50 euro
AMS reciprocity member	15 US$

The subscription fees listed above should be paid in euro by means of a cheque drawn on a bank in the Irish Republic, a Eurocheque, or an international money-order.

(3) The subscription fee for ordinary membership can also be paid in a currency other than euro using a cheque drawn on a foreign bank according to the following schedule:

If paid in United States currency then the subscription fee is US$ 30.00.
If paid in sterling then the subscription is $20.00.
If paid in any other currency then the subscription fee is the amount in that currency equivalent to US$ 30.00.
The amounts given in the table above have been set for the

current year to allow for bank charges and possible changes in exchange rates.

(4) Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

(5) Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.

(6) Subscriptions normally fall due on 1 February each year.

(7) Cheques should be made payable to the Irish Mathematical Society. If a Eurocheque is used then the card number should be written on the back of the cheque.

(8) Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

(9) Please send the completed application form with one year's subscription to:

The Treasurer, I.M.S.
Department of Mathematics
St Patrick's College
Drumcondra
Dublin 9, Ireland

James N. Flavin

1936 – 2012

Jim Flavin was born in Cork in December 3, 1936. He received his secondary education at Christian Brothers College, Cork where he had a distinguished academic career. He entered University College, Cork in 1954 and was awarded a First Class Honours Degree in Mathematical Science in 1957. He was awarded a M.Sc. Degree as well as the N.U.I. Travelling Studentship in 1959. Jim then embarked on his doctoral studies at King's College, Newcastle upon Tyne under the direction of Professor Albert Green and was awarded his Ph.D. degree from the University of Durham in 1962. The topic of his thesis was *Thermoelastic Wave Propagation in Prestressed Elastic Materials.* This was a profoundly stimulating time to be at King's College, since Professor Green's innovative work on fundamental continuum mechanics attracted several outstanding international visitors who influenced and enhanced an exciting research atmosphere. Jim took full advantage of the opportunities offered.

He was appointed Lecturer in Mathematical Physics at University College, Galway (now National University of Ireland, Galway) in 1962 and a year later at the remarkably young age of 26 was appointed Professor of Mathematical Physics and Head of Department in 1963. He held both positions until his retirement in 2002. He was the sole member of the Department until January 1965 and as a result carried an enormous teaching load.

At the time of Jim's appointment the College was governed by the University College Act 1929 which required the institution to provide academic programmes through the medium of the Irish language. While some members of the academic community took this requirement less than seriously, Jim understood it to be one of his conditions of appointment and undertook it extremely seriously. He realised that his secondary education had not fully equipped him adequately in Irish and set out to rectify this situation. He did this initially by frequently visiting the West Kerry Gaeltacht in Dunquin where Irish was the first language of the area. He also immersed himself in the study of Irish Language and Literature. His linguistic skills were such (he was also fluent in Italian and French) that he was soon regarded as an expert in the use of the Irish Language.

Jim's love of the Irish language was reflected in his enthusiasm for teaching courses in U.C.G. though the medium of Irish whenever possible. He was one of the few people who provided courses through Irish in the Science Faculty beyond first year level. He was for many years a member of a Department of Education working party on the development of an Irish terminology of scientific terms and he took particular pleasure when these efforts ultimately culminated in the publication of an English-Irish dictionary (Foclóir Eolaíochta, an Gúm, 1994).

He was an extremely active member of the Governing Body of U.C.G. for over twenty years until 1992. He was also a member of the Senate of The National University of Ireland for many years and served on the board of The School of Theoretical Physics of The Dublin Institute for Advanced Studies. Jim was also a member of National Committee for Theoretical and Applied Mechanics and a founder member of the Irish Mechanics Society. He was elected a Member of The Royal Irish Academy in 1999. He was also a Foreign Member of the Academia di Scienze Fisiche e Matematiche di Napoli.

Jim Flavin published throughout the years in various areas of Applied Mathematics. Initially his doctoral work led to a number of papers on wave propagation in elastic media. However he soon became interested in problems associated with classical elasticity. His initial work in this area was concerned with establishing bounds for the torsional rigidity of elastic cylinders of various cross sections and composed of a variety of types of elastic materials. His subsequent work on Saint-Venant's principle led to a series of papers frequently quoted by other leading scientists in the field. There followed a series of publications, mainly in conjunction with Robin Knops and Larry Payne on estimates for the asymptotic growth and decay of solutions in elasticity and other elliptic systems. In addition, he continued to be a sole author of several contributions devoted to diverse aspects of Mechanics

In 1993 Jim embarked on a collaboration with Salvatore Rionero of The University of Naples, which proved to be extraordinarily fruitful until Jim's death in April 2012. They spent many happy periods working together in both Galway and Naples. Their research was primarily concerned with the study of partial differential equations with particular emphasis on heat flow, nonlinear diffusion and Liapunov stability. Their numerous important and incisive results were combined into a well received book: *Quantitative Estimates for Partial Differential Equations: An Introduction,* 1995, CRC Press.

Jim had many interests. He was an extremely keen walker and swimmer. He played rugby while a student at Christian College, Cork. In recent years, primarily motivated by his son Aonghus, he developed a keen interest in soccer. He was an avid Liverpool supporter and was a regular visitor to Anfield. Jim's most recent visit was in January of this year, to see Liverpool play Manchester United.

Jim was a renowned story teller and seanchaí and is remember by his students as an excellent lecturer and communicator. He was a highly regarded member of the Irish and international mathematical communities and his death on April 13, 2012 leaves a void in the lives of those who knew him well. Above all he will be greatly missed by his wife Freda, his children Martina, Aonghus and Clíodhna and grandchildren.

Leaba imeasc na naomh go raibh aige.

Collected and compiled by Matt McCarthy, NUI Galway.

Irish Math. Soc. Bulletin
Number 69, Summer 2012, 9–9
ISSN 0791-5578

RANGES OF BIMODULE PROJECTIONS AND CONDITIONAL EXPECTATIONS

ROBERT PLUTA

This is an abstract of the PhD thesis *Ranges of Bimodule Projections and Conditional Expectations* written by R. Pluta under the supervision of Prof. Richard M. Timoney at the School of Mathematics, Trinity College Dublin and submitted in September 2011.

The algebraic theory of corner rings introduced by Lam [1] (as an abstraction of the properties of Peirce corners eRe of a ring R associated with an idempotent $e \in R$) is investigated in the context of C^*-algebras and operator algebras. The main result is as follows.

Theorem. *Let H be a Hilbert space with an orthonormal basis $(e_i)_{i\in I}$ (which may be countable or uncountable), and $\mathcal{B}(H)$ the algebra of bounded operators on H. Let $\mathcal{E}\colon \mathcal{B}(H) \to \mathcal{B}(H)$ be a linear map with range S a subalgebra such that $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$, $\mathcal{E}$ is an S-bimodule map, and $\mathcal{E}(x^*) = \mathcal{E}(x)^*$ for $x \in \mathcal{B}(H)$ ($\mathcal{E}$ is called a Lam conditional expectation). Then, if $e_i \otimes e_i^* \in S$ for $i \in I$, there is an equivalence relation on I such that $\mathcal{E}(x) = \sum_{j\in J} p_j x p_j$ for $x \in \mathcal{B}(H)$, where J is the set of equivalence classes, $p_j = \sum_{i\in j} e_i \otimes e_i^*$ for $j \in J$, and $e_i \otimes e_i^*$ is the operator that sends an element $h \in H$ to $\langle h, e_i\rangle e_i \in H$.*

This is generalized to purely atomic von Neumann algebras.

References

[1] T. Y. Lam: *Corner ring theory: a generalization of Peirce decompositions. I.* Algebras, rings and their representations, 153–182, World Sci. Publ., Hackensack, NJ, 2006.

School of Mathematics, Trinity College Dublin
E-mail address, R. Pluta: `plutar@tcd.ie`

2010 *Mathematics Subject Classification.* 46L05, 46L07.

Key words and phrases. C^*algebra, injective, noncommutative conditional expectation.

Received on 14-5-2012.

I would like to thank Prof. Richard M. Timoney for the care he took in supervising my PhD thesis.

©2012 Irish Mathematical Society

Irish Math. Soc. Bulletin
Number 69, Summer 2012, 11–23
ISSN 0791-5578

JOHN TODD AND THE DEVELOPMENT OF MODERN NUMERICAL ANALYSIS

NIALL MADDEN

Abstract. The purpose of this article is to mark the centenary of the birth of John Todd, a pioneer in the fields of numerical analysis and computational science. A brief account is given of his early life and career, and that of his wife, Olga Taussky, including experiences during World War II that led to him engaging with the then developing field of numerical analysis. Some of his contributions to the field, and the contexts in which they arose, are described.

1. Before the war

John (Jack) Todd's long and eventful life began on May 17th, 1911 in Carnacally, County Down. I give only an outline of these events here, and refer the interested reader to [1, 2, 5] for further details.

Having attended Methodist College in Belfast, Todd studied at Queen's University Belfast from 1928 to 1931, where A.C. Dixon was professor of Mathematics. He then went to Cambridge, but could not enrol for a bachelor's degree since he had not studied Latin, and so became a research student instead. He was supervised by J.E. Littlewood, who disapproved of the notion of doctoral degrees, so Todd never completed one. He worked under Littlewood's guidance on transfinite superpositions of absolutely continuous functions [25, 26].

He returned to lecture in Queen's University Belfast in 1933, working with J.G. Semple who had recently been appointed as professor. When Semple moved to King's College London in 1937, he invited

2010 *Mathematics Subject Classification.* 01-08, 01A60, 10A70, 65-XX.

Key words and phrases. John Todd, Olga Taussky, Biography, Numerical Analysis, Computational Mathematics.

Received on 14-5-2012; revised 29-5-2012.

The author is grateful to Prof. Tom Laffey for discussions about Olga Taussky and John Todd, to Frank Uhlig helpful comments and for the image in Figure 2, and to the Archives of the Mathematisches Forschungsinstitut Oberwolfach for permission to reproduce images in Figure 1.

©2012 Irish Mathematical Society

Todd to join him. Initially Todd taught aspects of real analysis there, particularly measure theory, but when another professor became ill while teaching a course in group theory, he was asked to take over.

He developed an interest in the area — enough to attend seminars on the topic at other London Colleges, and tackle a challenging research problem. That in turn led him to contact Olga Taussky who was at Westfield College, leading to a personal and professional partnership that was to last for nearly 60 years.

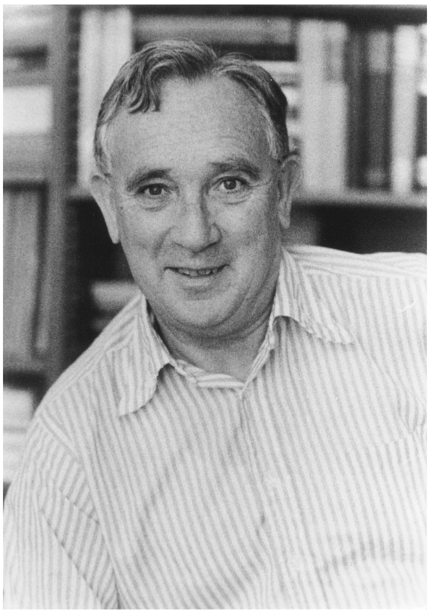

FIGURE 1. John Todd, 1977 and Olga Taussky, (circa 1932). (Source: Archives of the Mathematisches Forschungsinstitut Oberwolfach)

2. Taussky and Todd

Olga Taussky was a highly prolific and influential mathematician: she authored roughly 200 research articles and supervised the research of 14 graduate students; she was a founding editor of the journals *Linear Algebra and its Applications* and *Linear and Multilinear Algebra*; she received many awards and distinctions, including election as vice-president of the American Mathematical Society in 1985.

Taussky was born in 1906 in Olmütz, in the Austro-Hungarian empire. She studied in Vienna, initially majoring in both mathematics and chemistry, the latter being related to the family business.

However, she soon focused her attention on mathematics, eventually completing a doctorate in Vienna, under the supervision of Philip Furtwängler, a noted number theorist who contributed significantly to the development of class field theory.

In 1931 she moved to Göttingen, primarily to work with Richard Courant on editing Hilbert's papers on number theory, while also assisting Courant and Emmy Noether with their courses. However, the rise of antisemitism resulted in many academics in German universities, including Courant, Noether and Taussky, being forced to leave their positions. After a short time in Cambridge, Courant became a professor at New York University in 1936, while Noether moved to Bryn Mawr College in Pennsylvania. Taussky was awarded a three year research fellowship from Girton College in 1935, and decided to spend the first year of that in Bryn Mawr with Noether.

After applying for numerous positions, Olga Taussky was eventually appointed to a teaching post at Westfield College, a constituent of the University of London. In 1937 she met Todd and within a year they married, somewhat inauspiciously, on the day the Munich Agreement was signed. Their first joint articles, which show Taussky's emerging interest in the developing field of matrix theory (e.g., [21, 22]) were written in a bomb shelter in London. Their final joint paper [23], a historical note on links between the celebrated method of Cholesky and work of Otto Toeplitz, was completed shortly before Taussky's death in 1995.[1]

For more details of Taussky's life and career, see [17] and [47], and the autobiographical articles [24] and [20]: the former is primarily concerned with her life and experiences, the latter with her contributions to matrix theory. Her contributions to other areas of algebra are discussed in [16].

3. During the war

With the outbreak of World War II, and their colleges evacuated from London, Taussky and Todd moved to Belfast where they both

[1]The manuscript was originally submitted in 1995. Following the death of Taussky later that year, the manuscript was "lost" for several years. Some years after, there was renewed interest in the origins of Cholesky's method, including the discovery by Claude Brezinski of an original, unpublished, hand-written note by Cholesky describing it. So in 2005, Todd resubmitted the paper for publication in Numerical Algorithms.

taught for a year. Eventually they were to return to London and to work in scientific war jobs.

Taussky worked on aerodynamics at the National Physics Laboratory with the Ministry of Aircraft Production. Here she learned a great deal about differential equations, which had not interested her much previously, and matrix theory. She worked on problems in flutter [44], expressed as hyperbolic differential equations, and developed a technique that greatly reduced the amount of computational effort required to estimate the eigenvalues of certain matrices. Her idea was to use the simple, but very powerful, idea introduced by Geršgorin [12] which shows that the eigenvalues of an $n \times n$ matrix A with complex entries are contained in the union of n disks, where the i^{th} disk is centred on a_{ii} and has radius $\sum_{j \neq i} |a_{ij}|$. Taussky then used carefully chosen similarity transforms that reduce the radius of the disks, thus improving the accuracy of the estimate.

Although Geršgorin's work had received some attention, Taussky is often credited with popularising it, for example in [19]. Many generalisations and extensions were to follow, by Taussky and by others; an accessible account of these is given by Varga in [48]. The study of Geršgorin's disks were also a topic of research in the Ph.D. studies of Taussky's Irish student, Fergus Gaines [11].

While Taussky was working on aeronautics, Todd worked with the Admiralty in Portsmouth, initially on ways of counteracting acoustic mines. During that period he was struck by the amount of time that physicists spent doing routine calculations, while mathematicians were attempting to engage with engineering problems:

> "This was rather frustrating: physicists were doing elementary computing badly and mathematicians like me were trying to do physics. I thought that I could see a way to improve this mismatching" [1].

Todd persuaded his superiors to reassign him to London to establish what became the Admiralty Computing Service, centralising much of numerical computations for the naval service. He remained there until 1946.

In 1942 John von Neumann visited the Admiralty to inspect some of their ballistic facilities in connection with his work on developing the atomic bomb at Los Alamos. Todd was given the task of accompanying him, and introduced him to their computing facility.

This led to the rather remarkable claim, made by von Neumann, that Todd was responsible for getting him interested in computing!

Todd's work with the Admiralty also led to what he considered his greatest contribution to mathematics. In 1945 he was part of a small group that visited Germany to investigate mathematics that might be of interest to the Navy, such as the work of Konrad Zuse on programmable computers. The group also visited Oberwolfach, which they had heard was being used as a mathematics research centre. They arrived just in time to prevent it from being looted by Moroccan soldiers. Because it was in the French zone of occupation, Todd subsequently travelled to Paris to persuade the French government to maintain it. It later developed into a world renowned centre for mathematical research. For a lively recounting of the adventures of this time, see [41].

4. Conversion to Numerical Mathematics

Even before setting up the Admiralty Computing Service, Todd had developed an interest in the topic of computing, initially prompted by Alan Turing's work on computable numbers [45], and through contacts with the British Association for the Advancement of Science (BAAS), which was mainly concerned with making tables—often regarded as the primary goal of early scientific computing.

When he returned to Kings College in 1946, he taught the first course there on numerical mathematics. There were no text books for this developing area, so Todd developed his own notes. This included a section on the solution of systems of linear equations, featuring the method of Cholesky, which at the time was not well known in the mathematics literature.[2] It was through this course that Leslie Fox and colleagues at the Mathematics Division of the National Physics Laboratory became aware of this now standard method.

[2]André-Louis Cholesky was a French military officer. He developed his eponymous method for solving linear systems involving Hermitian, positive-definite matrices when he was engaged in computing solutions to certain least-squares problems that arise in geodesy. Compared to other approaches at the time, it was remarkably efficient—Cholesky reported that he could solve a system of 10 equations, to 5 decimal digits of accuracy, in under five hours! He explained his method to other topographers, but never published it. It was eventually published in a journal on geodesy by a colleague several years after Cholesky's untimely death towards the end of World War I [6]. See also [3].

Around this time, Artur Erdélyi and Todd wrote an article in *Nature* based on their observations of industrial mathematics research [9]. They argued for the foundation of an *Institute for Practical Mathematics* in the U.K. to provide instruction in the "mathematical technology" needed for developments in engineering, mathematical biology and economics. The call was not immediately heeded, but in 1947, Todd and Taussky moved to the United States, at the invitation of John Curtiss, to help establish the new Institute for Numerical Analysis at the National Bureau of Standards. Following an "inspirational" three months visiting von Neumann at Princeton [2], they began working at the INA, located at first at UCLA. They moved to Washington a year later where they stayed for 10 years.

In 1957 Todd and Taussky were offered positions at Caltech: John as Professor of Mathematics, and Olga as a research associate "with the permission but not the obligation to teach" [17].[3] Todd's appointment was made so as to develop courses in numerical analysis and computation within the mathematics department. They remained at Caltech for the rest of their lives. Taussky died on October 7th, 1995. Todd died on May 16th, 2007.

5. Todd and Numerical Analysis

Prior to the 1940s, a "computer" was usually understood to be a person who carried out calculations, and the designers of numerical algorithms had in mind the development of methods that were to be implemented by hand. Computing machinery was mainly limited to hand-operated mechanical calculators, such as the 10-digit Marchant Model 10 ACT—the first calculator used in Todd's course on numerical analysis at King's College in 1946. The same year, ENIAC (*Electronic Numerical Integrator And Computer*), the first general-purpose, programmable electronic computer, was launched.

From the 1940s, the rapid development of computer hardware was mirrored by rapid developments in the field of numerical analysis: it could be argued that the field emerged as a discipline in its own right between 1940 and 1960. (The term itself is usually credited to

[3]At the time, Caltech regulations prevented Todd and Taussky from holding professorships concurrently; this was only modified years later as a result of the recognition that Taussky was receiving as one of the country's leading mathematicians. Taussky was granted tenure in 1963 and a full professorship in 1971.

John Curtiss, and its first use was in the name of the Institute for Numerical Analysis that he founded in 1947). Landmark advances included the development of the Crank-Nicolson method for time-dependent partial differential equations in 1947, and the discovery of the more computationally efficient *alternating direction method* (ADI) by Peaceman and Rachford in 1953. The basis for Finite Element methods, now the most popular approach for numerical solution of partial differential equations in engineering applications, was provided by Courant in 1943, but their full potential was not realised until the 1960s. In 1947, von Neumann and Goldstine produced the first mathematical study of direct numerical solution of linear systems, with particular regard to the effects of round-off error.[4] New algorithms (and their analyses) for the iterative solution of linear systems included successive over-relaxation (SOR) in 1950, and the Conjugate Gradients method in 1952 (though the latter did not achieve significant popularity until much later). The *Fast Fourier Transform* of Cooley and Tukey was developed in 1965.

The American Mathematical Society's journal *Mathematical Tables and Other Aids to Computation* was launched in 1943, and renamed *Mathematics of Computation* in 1960. In 1959, the first journal for numerical analysis, Springer-Verlag's *Numerische Mathematik* was founded, with Robert Sauer, Alwin Walther, Eduard Stiefel, Alston Householder, and John Todd as editors. Todd served on the editorial board for 49 years. The SIAM Journal on Numerical Analysis was founded later, in 1964.

Readers interested in the history of the development of the field of numerical analysis in the 20$^{\text{th}}$ century should consult the recent article by Grcar [14] which pays special attention of the importance of von Neumann's work, particularly [49], and those who developed its ideas further, including John Todd.

5.1. **Articles.** Todd's first papers in numerical analysis were related to computational linear algebra, and the problems of ill-conditioning of matrices. Suppose A is a nonsingular matrix and we wish to solve the system $Ax = b$ by computational means. Simply representing b with finitely many digits introduces numerical error. In many cases

[4] Their paper [49] is often cited is the first in modern numerical analysis; others would give that honour to Turing for his 1936 paper on computable numbers [45], while in [38] Todd points to Comrie's 1946 article on the use of general business machines in solving computational problems [7].

of interest, the implementation of standard direct solution algorithm (all variants on Gaussian elimination) on a computer with finite precision greatly magnifies these errors. In [49], von Neumann and Goldstine introduced a quantity they called the "figure of merit" which gives an upper bound on the magnification of the errors. This is usually denoted $\kappa(A)$ and is now known as the *condition number* of the matrix, a term coined by Alan Turing in a related work written around the same time [46]. If $\kappa(A)$ is large, then the matrix is said to be *ill-conditioned*. It is usually defined as, for example,

$$\kappa(A) = \|A\|\|A^{-1}\|,$$

where $\|\cdot\|$ is one's favourite matrix norm, or as the ratio of the largest to smallest eigenvalue of A. This latter case can be useful in practice, since it does not require direct knowledge of A^{-1}. Furthermore, in [49] the "figure of merit" is given as $\|A\|_2\|A^{-1}\|_2 = \sigma_n/\sigma_1$ where $0 < \sigma_1 \leq \cdots \leq \sigma_n$ are the singular values of the invertible matrix A. But since in most of the cases considered, A is a symmetric positive definite matrix, this is the same as λ_n/λ_1 where $0 < \lambda_1 < \cdots < \lambda_n$ are the eigenvalues of A. Turing's proposed measures included using a scaled Frobenius norm. Todd [27, 29] studied this issue for a matrix arising in the numerical solution of a second-order elliptic problem in two variables by the standard finite difference method, and showed the relationship between several measures of conditioning proposed by Turing, von Neumann and others. His work was instrumental in Goldstine and von Neumann's quantity becoming accepted as *the* condition number. He went on to study fourth and higher-order problems in [32]. See [14] for a further discussion of this.

Other articles, including [28], are concerned with the stability of finite difference schemes, and propose that such analysis be done based on the matrix analysis of resulting linear systems. Similar ideas, but for several explicit and implicit schemes for time-dependent problems, are found in [34]. A mathematical analysis (explaining experimental results obtained by other authors) for an ADI method is given in [10]. The computation of special functions features in [15], for example, and the efficiency of methods for solving integral equations is considered in [35].

Although numerical analysis is often (and certainly, originally) understood as the mathematical study of computer algorithms for solving mathematical problems, in [33] Todd coined the term "ultramodern numerical analysis" (or "adventures with high speed automatic digital computing machines") which, unlike other areas of mathematics of the time, features aspects of experimentation, particularly where rigorous error analysis was not possible for sufficiently complicated problems. A systematic study of such experimentation, using matrix inversion as the main illustration, is given in [18]. An experimental study of a linear solver is given in [30], and of the application of a Monte Carlo method for solving a partial differential equation in [31] (as originally proposed by Courant, Friedrichs and Lewy in their celebrated 1928 paper that includes their famous condition for the stability of explicit schemes for time dependent problems [8]). Given the need to avoid over-extrapolation based on numerical experiments for a limited number of examples, Todd [33] cautioned that "separation of theoretical and applied numerical analysis is undesirable".

Although many of Todd's later papers were on the history of computational mathematics, he continued publishing original research into his seventies [4, 42] and eighties [43].

5.2. **Books.** While at the Bureau of Standards, Todd developed a programme to help train mathematicians in the new techniques of computational mathematics. He arranged for experts in the field to give courses in particular topics. At the suggestion of Taussky, the notes from these courses developed into the highly influential *Survey of Numerical Analysis* [36]. The first chapter, a reworking of [33] mentioned above, is titled *Motivation for Working in Numerical Analysis*, and notes that

> "the profession of numerical analysis is not yet so desirable that it is taken up by choice; indeed, although it is one of the oldest professions, it is only now becoming respectable".

He distinguished between what he termed classical, modern and ultramodern numerical analysis. Classical numerical analysis is concerned with solution, by hand, of problems in interpolation, integration, and the approximation of solutions to initial value problems[5]. Modern numerical analysis, on the other hand, is required for the exploitation of automatic digital computation. Finally, ultramodern numerical analysis relies on a combination of rigorous mathematics,

[5]See the fascinating monograph [13] for details on the fundamental developments of classical numerical analysis.

where problems permit it, and careful study of experimental results for problems whose complexity is beyond existing mathematical theory (for example, one might devise a numerical method for solving a nonlinear differential equation, give a mathematical analysis for a linearised variant and the results of supporting numerical experiments for the full nonlinear problem).

The survey contained contributions from, among others, Morris Newman, Harvey Cohen, Olga Taussky, Philip J. Davis, Werner Rheinbolt and Marshall Hall, Jr.. Topics covered include approximation of functions, the principles of programming, Turing Machines and undecidability, numerical linear algebra, differential equations, integral equations, functional analysis, block designs, number theory, and computational statistics.

Todd did not completely abandon his earliest research interests. His 1963 monograph, *Introduction to the Constructive Theory of Functions* [37] drew from the tradition of classical analysis to present sometimes neglected ideas on Chebyshev theory and orthogonal polynomials, while still providing "some mild propaganda for numerical analysis".

The courses in numerical analysis that Todd developed at Caltech became the basis for the two volume *Basic Numerical Mathematics*. As educational aspects of the subject developed, most presentations where aimed either at students at graduate or upper undergraduate level, or incorporated computational aspects into introductory courses in linear algebra and calculus. As he explains in Volume 1 [40], Todd aimed to introduce aspects of numerical computation after only the basics of calculus and algebra had been studied. He combined both "controlled numerical experiments", to reinforce ideas such as convergence and continuity, with "bad examples", to temper the tendency to rely on numerical experience rather then develop sound mathematical analyses:

> "The activities of the numerical analyst are similar to the highway patrol. The numerical analyst tries to prevent computational catastrophes".

Often, the existence of such "bad examples" is due to the subtle difference between real numbers and those that might be represented by computer. As an example (see [40, Chap. 3]) consider the divergent harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots .$$

If this sum is constructed on a computer it will appear to be convergent since, for large enough n, the computer will not be able to distinguish $1/n$ from zero.

Volume 1 [40], subtitled "Numerical Analysis", covers topics in interpolation, quadrature and difference equations, and are complemented by (relatively) elementary programming exercises. Since most programs required are for scalar problems, students were expected to develop a complete implementation of the algorithms themselves.

Volume 2 [39], "Numerical Algebra" deals with direct and iterative methods for solving linear systems of equations, and for the estimation of eigenvalues, with applications to curve fitting and solution of boundary value problems. Because the algorithms require the representation and storage of vectors and matrices, unlike the earlier volume, students were encouraged to use libraries of subroutines to complete programming assignments.

To summarise the importance, not only of these books, but of Todd's contribution in general, we give a quotation from A. S. Householder's review of [40]:

> "Probably no one has a practical and theoretical background surpassing that of the author, and this book is altogether unique".

References

[1] D. Albers. John Todd—numerical mathematics pioneer. *College Math. J.*, 38(1):2–23, 2007.

[2] F. L. Bauer. In memoriam John Todd (1911 – 2007). *Numer. Math.*, 108(1):1–6, 2007.

[3] C. Brezinski. The life and work of André Cholesky. *Numerical Algorithms*, 43(3):279–288, 2006.

[4] B. Carlson and J. Todd. The degenerating behavior of elliptic functions. *SIAM J. Numer. Anal.*, 20:1120–1129, 1983.

[5] S. K. Cohen. Interview with John Todd. *California Institute of Technology Archives Oral History Project*, 1997.

[6] Commandant Benoit. Note sur une méthode de réolution des équations normales provenant de l'application de la méthode des moindres carrés á un

FIGURE 2. John Todd and Olga Taussky-Todd, March 1973, Los Angeles. (Source: Frank Uhlig)

système d'équations linéaires en nombre inférieur à celui des inconnues.- Application de la méthode à la résolution d'un système défini d'équations linéaires (Procédé du Commandant Cholesky). *Bull. Géodésique*, 2:67–77, 1924.

[7] L. J. Comrie. Inverse Interpolation and Scientific Applications of the National Accounting Machine. *J. Roy. Statist. Soc.*, 3(2):87–114, 1936.

[8] R. Courant, K. Friedrichs, and H. Lewy. Über die partiellen Differenzengleichungen der mathematischen Physik. *Math. Ann.*, 100:32–74, 1928.

[9] A. Erdélyi and J. Todd. Advanced Instruction in Practical Mathematics. *Nature*, 158:690–692, 1946.

[10] D. Gaier and J. Todd. On the rate of convergence of optimal ADI processes. *Numer. Math.*, 9:452–459, 1966/1967.

[11] F. Gaines. *Some Generalizations of Commutativity for Linear Transformations on a Finite Dimensional Vector Space*. PhD thesis, California Institute of Technology, 1966.

[12] S. Geršgorin. Über die Abgrenzung der Eigenwerte einer Matrix. *Bull. Acad. Sc. Leningrad*, 1931:749–754, 1931.

[13] H. H. Goldstine. *A History of Numerical Analysis from the 16th through the 19th Century*. Springer-Verlag, New York, 1977. Studies in the History of Mathematics and Physical Sciences, Vol. 2.

[14] J. F. Grcar. John von Neumann's analysis of Gaussian elimination and the origins of modern numerical analysis. *SIAM Rev.*, 53(4):607–682, 2011.

[15] E. Kreyszig and J. Todd. The radius of univalence of the error function. *Numer. Math.*, 1:78–89, 1959.

[16] T. J. Laffey. Some aspects of Olga Taussky's work in algebra. *Linear Algebra Appl.*, 280(1):51–57, 1998. Special issue in memory of Olga Taussky-Todd.
[17] E. H. Luchins and M. A. McLoughlin. In memoriam: Olga Taussky-Todd [1906–1995]. *Notices Am. Math. Soc.*, 43(8):838–847, 1996.
[18] M. Newman and J. Todd. The evaluation of matrix inversion programs. *J. Soc. Ind. Appl. Math.*, 6:466–476, 1958.
[19] O. Taussky. A recurring theorem on determinants. *Amer. Math. Monthly*, 56:672–676, 1949.
[20] O. Taussky. How I became a torchbearer for matrix theory. *Amer. Math. Monthly*, 95(9):801–812, 1988.
[21] O. Taussky and J. Todd. A characterisation of algebraic numbers. *Proc. Roy. Irish Acad., Sect. A.*, 46:1–8, 1940.
[22] O. Taussky and J. Todd. Matrices with finite period. *Proc. Edinb. Math. Soc., II. Ser.*, 6:128–134, 1940.
[23] O. Taussky and J. Todd. Cholesky, Toeplitz and the triangular factorization of symmetric matrices. *Numer. Algorithms*, 41(2):197–202, 2006.
[24] O. Taussky-Todd. Autobiographical Essay. In D. Albers and G. Alexanderson, editors, *Mathematical People, Profiles and Interviews.* Birkhäuser Boston Inc., Boston, 1985.
[25] J. Todd. Superpositions of functions. I: Transfinite superpositions of absolutely continuous functions. *J. Lond. Math. Soc.*, 10:166–171, 1935.
[26] J. Todd. Superpositions of functions. II. Transfinite superpositions of absolutely continuous functions. *Proc. Lond. Math. Soc., II. Ser.*, 41:433–439, 1936.
[27] J. Todd. The condition of certain matrices. I. *Quart. J. Mech. Appl. Math.*, 2:469–472, 1949.
[28] J. Todd. Notes on modern numerical analysis. I. Solution of differential equations by recurrence relations. *Math. Tables and Other Aids to Computation*, 4:39–44, 1950.
[29] J. Todd. The condition of a certain matrix. *Proc. Camb. Philos. Soc.*, 46:116–118, 1950.
[30] J. Todd. Experiments on the inversion of a 16×16 matrix. *Nat. Bur. Standards, Appl. Math. Ser.*, 29:113–115, 1953.
[31] J. Todd. Experiments in the solution of differential equations by Monte Carlo methods. *J. Washington Acad. Sci.*, 44:377–381, 1954.
[32] J. Todd. The condition of certain matrices. II. *Arch. Math.*, 5:249–257, 1954.
[33] J. Todd. Motivation for working in numerical analysis. *Commun. Pure Appl. Math.*, 8:97–116, 1955.
[34] J. Todd. A direct approach to the problem of stability in the numerical solution of partial differential equations. *Commun. Pure Appl. Math.*, 9:597–612, 1956.
[35] J. Todd. The condition of certain integral equations. In *Symposium on the numerical treatment of ordinary differential equations, integral and integro-differential equations (Rome, 1960)*, pages 306–311. Birkhäuser, Basel, 1960.
[36] J. Todd, editor. *Survey of Numerical Analysis.* New York-San Francisco-Toronto-London: McGraw-Hill Book Company, Inc. XVI, 589 p. , 1962.
[37] J. Todd. *Introduction to the Constructive Theory of Functions.* Academic Press Inc., New York, 1963.

[38] J. Todd. Numerical analysis at the National Bureau of Standards. *SIAM Rev.*, 17:361–370, 1975.

[39] J. Todd. *Basic Numerical Mathematics. Vol. 2: Numerical Algebra.* International Series of Numerical Mathematics. Vol. 22. Basel - Stuttgart: Birkhäuser Verlag. 216 p., 1977.

[40] J. Todd. *Basic Numerical Mathematics. Vol. 1: Numerical Analysis.* International Series of Numerical Mathematics (ISNM), Vol. 14. Basel, Stuttgart: Birkhäuser Verlag. 253 p., 1979.

[41] J. Todd. Oberwolfach - 1945. In E. F. Beckenback and W. Walter, editors, *General Inequalities 3, 3rd int. Conf., Oberwolfach 1981, ISNM 64*, pages 19–22. Birkhäuser Verlag, 1983.

[42] J. Todd. The Weierstrass mean. I: The periods of $\wp(z|e_1, e_2, e_3)$. *Numer. Math.*, 57(8):737–746, 1990.

[43] J. Todd and R. S. Varga. Hardy's inequality and ultrametric matrices. *Linear Algebra Appl.*, 302/303:33–43, 1999. Special issue dedicated to Hans Schneider (Madison, WI, 1998).

[44] O. Todd. *On some boundary value problems in the theory of the non-uniform supersonic motion of an aerofoil.* Rep. and Memoranda no. 2141(8961). Ministry of Supply [London], Aeronaut. Res. Council, 1945.

[45] A. Turing. On computable numbers, with an application to the Entscheidungsproblem. *Proc. Lond. Math. Soc., II. Ser.*, 42:230–265, 1936.

[46] A. M. Turing. Rounding-off errors in matrix processes. *Quart. J. Mech. Appl. Math.*, 1:287–308, 1948.

[47] F. Uhlig. Olga Taussky, a torchbearer for mathematics and teacher of mathematicians. *PAMM*, 7(1):1170501–1170503, 2007.

[48] R. S. Varga. *Geršgorin and his circles*, volume 36 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2004.

[49] J. von Neumann and H. H. Goldstine. Numerical inverting of matrices of high order. *Bull. Amer. Math. Soc.*, 53:1021–1099, 1947.

Niall Madden. Lecturer in Mathematics at NUI Galway. His research interests are in the area of numerical analysis, particularly in developing finite difference and finite element schemes for solving boundary value problems.

School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway

E-mail address: `Niall.Madden@NUIGalway.ie`

Irish Math. Soc. Bulletin
Number 69, Summer 2012, 25–32
ISSN 0791-5578

"ODD" MATRICES AND EIGENVALUE ACCURACY

DAVID JUDGE

Abstract. A definition of *even* and *odd* matrices is given, and some of their elementary properties stated. The basic result is that if λ is an eigenvalue of an odd matrix, then so is $-\lambda$. Starting from this, there is a consideration of some ways of using odd matrices to test the order of accuracy of eigenvalue routines.

1. Definitions and some elementary properties

Let us call a matrix W **even** if its elements are zero unless the sum of the indices is even – i.e. $W_{ij} = 0$ unless $i + j$ is even; and let us call a matrix B **odd** if its elements are zero unless the sum of the indices is odd – i.e. $B_{ij} = 0$ unless $i + j$ is odd. The non-zero elements of W and B (the letters W and B will always denote here an even and an odd matrix, respectively) can be visualised as lying on the white and black squares, respectively, of a chess board (which has a white square at the top left-hand corner).

Obviously, any matrix A can be written as $W + B$; we term W and B the *even* and *odd* parts, respectively, of A. Under multiplication, even and odd matrices have the properties, similar to those of even and odd functions, that

$$\begin{array}{ccccc} \textit{even} \times \textit{even} & \text{and} & \textit{odd} \times \textit{odd} & \text{are} & \textit{even}, \\ \textit{even} \times \textit{odd} & \text{and} & \textit{odd} \times \textit{even} & \text{are} & \textit{odd}. \end{array}$$

From now on, we consider only **square** matrices. It is easily seen that, if it exists, the inverse of an even matrix is even, the inverse of an odd matrix is odd. It is also easily seen that in the PLU decomposition of a non-singular matrix A which is either even or odd, L and U are always even, while P is even or odd according as A is.

2010 *Mathematics Subject Classification.* 15A18,15A99,65F15.
Key words and phrases. even matrix, odd matrix, eigenvalues, test matrix.
Received on 27-4-2012; revised 11-7-2012.
The author would like to express his gratitude for helpful discussions with T. Laffey, D. Lewis, and the late Fergus Gaines.

©2012 Irish Mathematical Society

We note also that in the QR factorisation of a non-singular even matrix, R and Q are both even, while for an odd matrix, R is even, Q is odd. (This is most easily seen by viewing the QR factorisation in its Gram-Schmidt orthogonalisation aspect.) Thus the property of being even or odd is preserved under the basic QR algorithm for eigenvalues: however, for odd matrices, this is not true for the QR algorithm with shifts.

2. The basic result

The following very elementary result is basic:

Proposition 2.1. *If λ is an eigenvalue of an odd matrix B, then so is $-\lambda$.*

This can be seen easily in two different ways.

First Proof. If $D_{jk} = (-1)^k \delta_{jk}$, $\quad D^{-1}(W + B)D = W - B$, so that $W + B$, $W - B$ have the same eigenvalues, for arbitrary square (even and odd) matrices W and B. Putting $W = 0$, the result follows. $\square$

Second Proof. We can write any vector $\mathbf{x}$ as $\mathbf{x} = \mathbf{u} + \mathbf{v}$, where $u_i = 0$, i odd, $v_i = 0$, i even. We call $\mathbf{u}$ and $\mathbf{v}$ *even* and *odd* vectors, respectively, and refer to them as the *even* and *odd* parts of $\mathbf{x}$. (We assume that indices run from 1 to n, the order of the matrix. However, choosing the index origin as 0 merely interchanges the values of $\mathbf{u}$ and $\mathbf{v}$, and makes no difference to what follows.) We note that

$$\textit{if } B \textit{ is odd}, \quad \textit{then} \quad B\mathbf{u} \ \textit{is odd}, \quad B\mathbf{v} \ \textit{is even}. \tag{1}$$

Now if $\mathbf{x}$ is an eigenvector of B with eigenvalue λ,

$$B\mathbf{x} = \lambda \mathbf{x}, \tag{2}$$

writing $\mathbf{x} = \mathbf{u} + \mathbf{v}$, and using (1), we must have

$$B\mathbf{u} = \lambda \mathbf{v} \ , \qquad B\mathbf{v} = \lambda \mathbf{u}. \tag{3}$$

Then

$$B(\mathbf{u} - \mathbf{v}) = B\mathbf{u} - B\mathbf{v} = \lambda \mathbf{v} - \lambda \mathbf{u} = -\lambda(\mathbf{u} - \mathbf{v}), \tag{4}$$

so that $-\lambda$ is an eigenvalue of B, with eigenvector $\mathbf{u} - \mathbf{v}$. $\square$

This property is useful in giving a simple indication of the accuracy of eigenvalue routines. For a completely arbitrary odd matrix B, random or otherwise, with real or complex elements, the non-zero eigenvalues can be sorted into pairs $(\lambda_i^1, \lambda_i^2)$ such that

$$\lambda_i^1 + \lambda_i^2 = 0. \tag{5}$$

(As far as this property is concerned, there is no initial error whatever in entering B .) If the computed values, denoted by $\hat{\lambda}$, are sorted into corresponding pairs $(\hat{\lambda}_i^1, \hat{\lambda}_i^2)$, and

$$\delta_i \equiv \hat{\lambda}_i^1 + \hat{\lambda}_i^2 , \tag{6}$$

then $\max|\delta_i|$ gives an estimate of the error.

A check on the accuracy of the eigenvectors is also possible. The argument of proof 2 shows that if $\mathbf{x} = \mathbf{u} + \mathbf{v}$ is an eigenvector of B corresponding to a simple eigenvalue λ , and if $\mathbf{y}$ is an eigenvector corresponding to $-\lambda$, then $\mathbf{y}$ is of the form $\mathbf{y} = \theta\mathbf{u} + \phi\mathbf{v}$, with the *same* $\mathbf{u}$ and $\mathbf{v}$, where θ and ϕ are scalars. (The fact that $\phi = -\theta$ here is not essential.) Thus if we multiply $\mathbf{y}$ by a factor to make one of its even components (say the first non-zero one, or the largest one) equal to the corresponding component of $\mathbf{x}$, producing $\mathbf{z}$, say, we must have $\mathbf{z} = \mathbf{u} + \rho\mathbf{v}$, where ρ is some scalar. When we subtract $\mathbf{z}$ from $\mathbf{x}$, all the other even components must also cancel exactly. Doing this for the computed vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, the actual deviations from zero of these components give an accuracy estimate. The odd components can be checked similarly.

3. Even matrices

For even matrices, there is no necessary relation whatever between any of the eigenvalues – e.g., a diagonal matrix with arbitrary values is even.

However, it is of interest to consider a further subdivision of such matrices into a *doubly-even* (in brief, d-even) part and a *doubly-odd* (in brief, d-odd) part,

$$W = W_{ee} + W_{oo}, \tag{7}$$

where $(W_{ee})_{ij} = 0$ unless i and j are both *even*, $(W_{oo})_{ij} = 0$ unless i and j are both *odd*.

Under multiplication, doubly-even and doubly-odd matrices have the properties that

$$\begin{array}{llllll} \textit{d-even} \times \textit{d-even} & \text{is} & \textit{d-even}, & \textit{d-odd} \times \textit{d-odd} & \text{is} & \textit{d-odd}, \\ \textit{d-even} \times \textit{d-odd} & \text{is} & \textit{zero}, & \textit{d-odd} \times \textit{d-even} & \text{is} & \textit{zero}. \end{array}$$

Thus the doubly-even and doubly-odd parts are completely 'uncoupled' under multiplication.

If I denotes the unit matrix, let I_{ee} and I_{oo} denote its doubly-even and doubly-odd parts, so that

$$I = I_{ee} + I_{oo}. \tag{8}$$

We can now see that the d-even and d-odd parts are also 'uncoupled' when taking inverses, in the following sense. If W is non-singular, let X and Y denote the d-even and d-odd parts, respectively, of W^{-1}. Then

$$X \times W_{ee} = I_{ee}, \quad X \times W_{oo} = 0, \quad Y \times W_{ee} = 0, \quad Y \times W_{oo} = I_{oo}.$$

The eigenvalue problem for W also splits into two completely independent sub-problems:

$$W\mathbf{x} = \lambda\mathbf{x} \quad \Longrightarrow \quad W_{ee}\mathbf{u} = \lambda\mathbf{u}, \; W_{oo}\mathbf{v} = \lambda\mathbf{v},$$

where $\mathbf{u}$ and $\mathbf{v}$ are the even and odd parts of $\mathbf{x}$.

These properties of even matrices, trivial to check, are even more obvious on noting that under a simple reordering transformation, W is equivalent to a block-diagonal matrix of the form

$$T = \left[\begin{array}{c|c} T1 & 0 \\ \hline 0 & T2 \end{array}\right] \tag{9}$$

where $T1$, of dimension $\lfloor (n+1)/2 \rfloor$, contains the non-zero elements of W_{oo}, and $T2$, of dimension $\lfloor n/2 \rfloor$, contains those of W_{ee}.

We note that under the same transformation, an odd matrix B is similarly equivalent to a skew-diagonal block matrix

$$S = \left[\begin{array}{c|c} 0 & S1 \\ \hline S2 & 0 \end{array}\right]. \tag{10}$$

Thus the determinant of an even matrix always factorizes,

$$\det(W) = \det(T) = \det(T1).\det(T2),$$

and the determinant of an odd matrix either factorizes or is zero:

$$\begin{aligned} \det(B) = \det(S) &= (-1)^{n/2}\det(S1).\det(S2), \text{ if } n \text{ is even}, \\ &= 0 \text{ if } n \text{ is odd} \end{aligned}$$

(since if n is odd, $S1$ and $S2$ are not square). This latter gives one proof of the almost obvious fact that an odd matrix of odd dimension always has the eigenvalue $\lambda = 0$.

4. Further eigenvalue matchings

We consider now the matrix

$$M \;\equiv\; \alpha I_{ee} + \beta I_{oo} + B \tag{11}$$

where α and β are scalars, B as usual denotes an odd matrix, and I_{ee} and I_{oo} are as defined above at (8). If λ is an eigenvalue of M, the eigenvalue equation

$$M\mathbf{x} = \lambda\mathbf{x} \tag{12}$$

can be written as the pair of equations

$$\alpha\mathbf{u} + B\mathbf{v} = \lambda\mathbf{u}, \tag{13}$$

$$B\mathbf{u} + \beta\mathbf{v} = \lambda\mathbf{v} \tag{14}$$

where $\mathbf{u}$ and $\mathbf{v}$ are the even and odd parts of $\mathbf{x}$. We can now proceed in two different ways.

First, on defining $\mathbf{y}$ by

$$\mathbf{y} = (\lambda - \alpha)\mathbf{u} + (\beta - \lambda)\mathbf{v} \tag{15}$$

we find that

$$M\mathbf{y} = (\alpha + \beta - \lambda)\mathbf{y}. \tag{16}$$

Thus the eigenvalues of the matrix M can be grouped into pairs (λ_1, λ_2) such that

$$\lambda_1 + \lambda_2 = \alpha + \beta. \tag{17}$$

This may be viewed as a generalisation of the basic result for a pure odd matrix B, where $\alpha = \beta = 0$.

Second, on defining $\mathbf{z}$ by

$$\mathbf{z} = \sqrt{(\lambda - \alpha)}\;\mathbf{u} + \sqrt{(\lambda - \beta)}\;\mathbf{v}, \tag{18}$$

we find that

$$B\mathbf{z} = \sqrt{\lambda - \alpha}\;\sqrt{\lambda - \beta}\;\mathbf{z}. \tag{19}$$

Thus for each eigenvalue λ of M, there corresponds an eigenvalue κ of B such that

$$\kappa = \sqrt{(\lambda - \alpha)(\lambda - \beta)}. \tag{20}$$

It should be emphasised that the relation (17) relates two eigenvalues of the *same* matrix, generated during *one* calculation, while

(20) relates eigenvalues of two *different* matrices, generated in *two independent* calculations.

Based on (20), we can give now an error estimate alternative to (6). It is convenient to take $\beta = -\alpha$, and solve for $\lambda, \lambda = \sqrt{(\kappa^2 + \alpha^2)}$, so that the sets of λ's and κ's can be sorted into pairs (λ_i, κ_i) such that

$$\lambda_i - \sqrt{(\kappa_i^2 + \alpha^2)} = 0. \tag{21}$$

If the computed values, denoted by $\hat{\lambda}, \hat{\kappa}$, are sorted into corresponding pairs $(\hat{\lambda}_i, \hat{\kappa}_i)$, and

$$\delta_i \equiv \hat{\lambda}_i - \sqrt{(\hat{\kappa}_i^2 + \alpha^2)}\ , \tag{22}$$

then again $\max|\delta_i|$ gives an estimate of the error.

We note that for any estimate based on either (17) or (as here) (20), one can check on the eigenvectors also, just as in Section 2.

5. Discussion

If one is using a package to calculate eigenvalues and wishes to get some idea of the accuracy of the results, it is natural to see how it performs on *test matrices.* An ideal test matrix is one which can be entered exactly, and whose eigenvalues are known in advance, either exactly or to very high precision. The error level can thus be assessed directly by observing the differences between the computed values and the correct values.

Rather than importing samples of this type, one may prefer to use something which is easily generated. For example, a simple method is to choose a set of numbers and form a diagonal matrix D, say, with these values; then choose an arbirary non-singular matrix X, say, and take $M \equiv XDX^{-1}$ as the test matrix. Its eigenvalues should be the chosen numbers, and its eigenvectors the columns of X. However, there is a snag: what is entered into the eigenvalue calculation is not, in fact, the exact 'known-eigenvalue' matrix M, but a computed approximation, $\hat{M}$, say; and so one cannot tell how much of the observed error is due to the eigenvalue calculation, and how much arises in computing $\hat{M}$.

An alternative approach, the one adopted here, is to use an exact-entry test matrix, or pair of matrices, whose eigenvalues are *not* known in advance: but, instead, some *relation* which should hold between these values is known. The extent to which the the computed values fail to satisfy this relation gives an estimate of the error due to the eigenvalue calculation alone.

However, caution is needed with this approach. Consider the following argument: "if λ is an eigenvalue of a *real* matrix, then so is $\bar{\lambda}$. Therefore we can sort the complex eigenvalues into pairs, the sum of whose imaginary parts should be zero. The difference from zero of the sum of the computed imaginary parts then gives an indication of the error of these computed values." This is, of course, completely erroneous, because the imaginary parts will be evaluated at the *same* point of the calculation as being $\pm$ the square-root of the *same* number,Q, say; and their sum will thus cancel perfectly, even though Q may differ widely from the correct value.

This shows that we must guard against the possibilty that the supposed 'accuracy-checking' relation between eigenvalues may be automatically satisfied at the time of their calculation. (One might worry that this could somehow be happening in (6), i.e. that the the 'equal-and-opposite' pairs might be produced in some correlated way because their magnitudes are equal. To circumvent this, one could add α times the unit matrix to B before finding the eigenvalues, and then subtract α from each before pairing - equivalent to (17) with $\beta = \alpha$. However, trials show that this step is unnecessary.) Clearly, there is no possiblity whatever of correlation between the errors of the $\kappa's$ and $\lambda's$ in (22).

For a *real* odd matrix B, if λ is an eigenvalue, then $\pm\lambda, \pm\bar{\lambda}$ are all in the list of eigenvalues, which complicates the sorting and matching of the imaginary parts. To keep (6) easy to implement, one can consider just the real parts, and simply ignore the imaginary parts of the λ's, implicitly assuming that the errors of the real and imaginary parts are of the same order. (This may be false in special circumstances, e.g. if B is antisymmetric – a case where, in fact, using (6) fails completely.) Using (22) has no such problems: we can take all the real parts and all the imaginary parts, of both the $\hat{\lambda}_i$ and the $\sqrt{(\hat{\kappa}_i^2 + \alpha^2)}$, as positive, and sort these real and imaginary parts independently.

In trials of random odd matrices (real and complex) of dimension up to 500, using (6)and (22) gave results of the same order, differing by a factor often close to one, and rarely exceeding two.

6. Addendum

The author is grateful to the referee for pointing out that very similar concepts have been introduced and used by Liu *et al* in the field of computer science: in [1], a square matrix A of even dimension n is decomposed into four sub-matrices, each of dimension $n/2$, containing those elements A_{ij} of A for which i and j are *even* and *even*, *even* and *odd*, *odd* and *even*, or *odd* and *odd*, respectively. (These correspond to the T_1, S_1, S_2, and T_2 of section 3, the index origin in [1] being 0.) These authors then exploit this partitioning to break down the process of evaluating $A \times \mathbf{b}$, where $\mathbf{b}$ is a vector of dimension n, into four processes involving the even and odd sub-vectors of $\mathbf{b}$ (each of dimension $n/2$), as a key step in designing an improved hardware module for use in vector digital signal processing.

If, motivated by this work, we consider a splitting, similar to 7, of an n-dimensional square odd matrix B into an "*even-odd*" part and an "*odd-even*" part (both n-dimensional),

$$B = B_{eo} + B_{oe}, \tag{23}$$

where $(B_{eo})_{ij} = 0$ unless i is *even*, j is *odd*, $(B_{oe})_{ij} = 0$ unless i is *odd*, j is *even*, it is of interest to note the easily-proved facts

(a) if λ is an eigenvalue of $B_{eo} + B_{oe}$, then $i\lambda$ is an eigenvalue of $B_{eo} - B_{oe}$,

(b) if W is an even matrix, the matrices $W, W + B_{eo}$ and $W + B_{oe}$ all have the same eigenvalues.

References

[1] Liu et al.: *Matrix odd-even partition: a high power-efficient solution to the small grain data shuffle.* Proc of the 6th IEEE International Conf on Networking, Architecture, and Storage. (2011) 348-54. (DOI 10.1109/NAS.2011.29)

David Judge was a member of the Mathematical Physics Department, University College Dublin, for many years until his retirement in 1996.

Mathematical Physics Department, University College Dublin
E-mail address: `david.judge@ucd.ie`

Irish Math. Soc. Bulletin
Number 69, Summer 2012, 33–46
ISSN 0791-5578

SPECTRAL PERMANENCE

ROBIN HARTE

Abstract. Several kinds of generalized inverse bounce off one another in the proof of a variant of spectral permanence for C* embeddings.

This represents an expanded version of our talk to the IMS meeting of August 2012, which in turn was based on the work [3] of Dragan Djordjevic and Szezena Zivkovic of Nis, in Serbia.

1. Gelfand property

Spectral permanence, for C* algebras, says that the spectrum of an element $a \in A \subseteq B$ of a C* algebra is the same whether it is taken relative to the subalgebra A or the whole algebra B: this discussion is sparked by the effort to prove that the same is true of a variant of spectral permanence in which the two-sided inverse, whose presence or not defines "spectrum", is replaced by a *generalized inverse*. The argument involves a circuitous tour through "group inverses", "Koliha-Drazin inverses" and "Moore-Penrose inverses"; it turns out that the induced variants of spectral permanence are curiously inter-related.

Suppose $T : A \to B$ is a *semigroup homomorphism*, where we insist that a semigroup A has an identity 1 , and that a homomorphism $T : A \to B$ respects that: we might indeed talk about a *functor* between categories. It then follows, writing A^{-1} for the invertible group in A, that

$$T(A^{-1}) \subseteq B^{-1} , \tag{1.1}$$

or equivalently, turning it inside out,

$$A^{-1} \subseteq T^{-1}B^{-1} . \tag{1.2}$$

2010 *Mathematics Subject Classification.* 46H05, 47A05, 47A53.
Key words and phrases. Semigroup homomorphisms, spectral permanence, generalized permanence, simply polar elements.
Received on 10-3-2011; revised 5-8-2012.

©2012 Irish Mathematical Society

At its most abstract then "spectral permanence" for the homomorphism T says that (1.2) holds with equality:

$$T^{-1}B^{-1} \subseteq A^{-1} \ . \tag{1.3}$$

In words, it is tempting to describe (1.3) by saying "Fredholm implies invertible". We shall also describe (1.3) as the *Gelfand property*, since it also holds, famously, when

$$T = \Gamma : A \to C(X) \subseteq \mathbf{C}^X \tag{1.4}$$

is the *Gelfand representation* of a commutative Banach algebra A; here of course $X = \sigma(A)$ is the "maximal ideal space" of the algebra A. We might notice a secondary instance of spectral permanence in the embedding

$$C(X) \subseteq \mathbf{C}^X \tag{1.5}$$

of continuous functions among arbitrary functions; similarly, for a Banach space X, the embedding

$$B(X) \subseteq L(X) \tag{1.6}$$

of bounded operators among arbitrary linear operators has spectral permanence, but only thanks to the ministrations of the *open mapping theorem*. Another elementary example is the *left regular representation*

$$L : A \to A^A \tag{1.7}$$

of the semigroup A as mappings, where, for $a \in A$,

$$L_a(x) = ax \ (x \in A) \ . \tag{1.8}$$

Less familiar is a *commutant embedding*

$$J : A = \mathrm{comm}_B(K) \to B \ , \tag{1.9}$$

where

$$\mathrm{comm}_B(K) = \{b \in B : a \in K \Longrightarrow ba = ab\} \tag{1.10}$$

and of course $J(a) = a$: here spectral permanence reflects the fact that two-sided inverses double commute:

$$a \in B^{-1} \Longrightarrow a^{-1} \in \mathrm{comm}_B^2(a) \ . \tag{1.11}$$

If in particular the semigroup A is a ring, having therefore a background "addition" and a distributive law, then we can quotient out the *Jacobson radical*

$$\mathrm{Rad}(A) = \{a \in A : 1 - Aa \subseteq A^{-1}\} \ , \tag{1.12}$$

in which every possible expression $1 - ca$ has an inverse: now it is easily checked that

$$K : a \mapsto a + \mathrm{Rad}(A)\ (A \to A/\mathrm{Rad}(A)) \tag{1.13}$$

has spectral permanence. Our final example will be the most familiar, if not by any means the most elementary: it is the *determinant*

$$\det : \mathbf{C}^{n\times n} \to \mathbf{C}\ , \tag{1.14}$$

which indeed "determines" whether or not a square matrix is invertible.

2. Spectral permanence

Mathematicians are thus prepared to go to a lot of trouble to establish spectral permanence. If we specialise to linear homomorphisms between (complex) linear algebras then we meet the phenomenon of *spectrum*, defining for each $a \in A$,

$$\sigma_A(a) = \{\lambda \in \mathbf{C} : a - \lambda \notin A^{-1}\}\ ; \tag{2.1}$$

the idea is to harness complex analysis to the theory of invertibility. Now we can rewrite (1.1) to say that, for arbitrary $a \in A$,

$$\sigma_B(Ta) \subseteq \sigma_A(a)\ , \tag{2.2}$$

while the Gelfand property (1.3) says that (2.2) holds with equality, giving indeed "spectral permanence".

If we specialise to isometric Banach algebra homomorphisms then there is built in a certain degree of spectral permanence, to the extent that we always get

$$\partial\sigma_A(a) \subseteq \sigma_B(Ta)\ \ : \tag{2.3}$$

the topological boundary of the larger spectrum is included in the smaller. Equivalently, it turns out, this means that

$$\sigma_A(a) \subseteq \eta\sigma_B(Ta)\ , \tag{2.4}$$

where the *connected hull* ηK of a compact subset $K \subseteq \mathbf{C}$ is the complement of the unbounded connected component of the complement $\mathbf{C} \setminus K$. This has spin-off: if for a particular element $a \in A$ either the larger spectrum is all boundary,

$$\sigma_A(a) \subseteq \partial\sigma_A(a)\ , \tag{2.5}$$

or the smaller spectrum fills out its connected hull,

$$\eta\sigma_B(Ta) \subseteq \sigma_B(Ta)\ , \tag{2.6}$$

then the homomorphism $T : A \to B$ has "spectral permanence at" $a \in A$, in the sense of equality in (2.2). This holds if for example the spectrum is either *real* or *finite*.

If more generally the homomorphism $T : A \to B$ is one-one there is at least inclusion

$$\text{iso}\ \sigma_A(a) \subseteq \sigma_B(Ta)\ . \tag{2.7}$$

3. Generalized permanence

If A is a semigroup we shall write

$$A^\cap = \{a \in A : a \in aAa\} \tag{3.1}$$

for the "regular" or *relatively regular* elements of A, those $a \in A$ which have a *generalized inverse* $c \in A$ for which

$$a = aca\ \ : \tag{3.2}$$

we remark that if (3.2) holds the products

$$p = ca = p^2\ ,\ q = ac = q^2 \tag{3.3}$$

are both *idempotent*. Generally if $T : A \to B$ is a homomorphism there is inclusion

$$T(A^\cap) \subseteq B^\cap \subseteq B\ , \tag{3.4}$$

and hence also

$$A^\cap \subseteq T^{-1}(B^\cap) \subseteq A\ . \tag{3.5}$$

If there is equality in (3.4) we shall say that T has *generalized permanence*. This happens for example when

$$T^{-1}(0) \subseteq A^\cap\ ,\ T(A) = B\ \ : \tag{3.6}$$

recall the implication

$$(a - aAa) \cap A^\cap \neq \emptyset \Longrightarrow a \in A^\cap\ . \tag{3.7}$$

This does not however happen when T is quotienting out the radical as in (1.10), unless the ring A is *semi simple*: for notice

$$\text{Rad}(A) \cap A^\cap = \{0\}\ . \tag{3.8}$$

It follows that spectral permanence is not in general sufficient for generalized permanence. Indeed by (3.8) spectral and generalized permanence together imply that a homomorphism $T : A \to B$ is one one; further (1.5) shows that spectral permanence and one one do not together imply generalized permanence. If A is the ring of continuous homomorphisms $a : X \to X$ on a Hausdorff topological abelian group X then it is necessary for $a \in A^\cap$ that a have *closed range*

$$a(X) = \mathrm{cl}\ a(X)\ : \tag{3.9}$$

this is because

$$a(X) = ac(X) = (1 - ac)^{-1}(0) \tag{3.10}$$

is the null space of the complementary idempotent. Thus the embedding (1.6) is another example with spectral but not generalized permanence.

4. Simple permanence

If in particular there is $c \in A$ for which

$$a - aca = 0 = ac - ca\ , \tag{4.1}$$

then $a \in A$ is very special; this happens if $a \in A$ is either invertible, or idempotent, or more generally the commuting product of an invertible and an idempotent. When (4.1) holds we shall say that $a \in A$ is *simply polar*: in Banach-algebra-land $0 \in \mathbf{C}$ can be at worst a simple pole of the *resolvent mapping*

$$(z - a)^{-1} : \mathbf{C} \setminus \sigma(a) \to A\ . \tag{4.2}$$

In the group theory world the product cac is referred to as the *group inverse* for $a \in A$. We remark that it is necessary and sufficient for $a \in A$ to be simply polar that

$$a \in a^2 A \cap Aa^2\ : \tag{4.3}$$

indeed [15],[19],[20] there is implication

$$a^2 u = a = va^2 \Longrightarrow au = va\ ,\ aua = a = ava\ , \tag{4.4}$$

giving (4.1) with $c = vau$.

We shall write $\mathrm{SP}(A)$ for the simply polar elements of a semigroup A and observe, for homomorphisms $T : A \to B$, that

$$T\ \mathrm{SP}(A) \subseteq \mathrm{SP}(B) \subseteq B\ , \tag{4.5}$$

and hence

$$\mathrm{SP}(A) \subseteq T^{-1}\mathrm{SP}(B) \subseteq A\ ; \tag{4.6}$$

when there is equality in (4.5) we shall say that $T : A \to B$ has *simple permanence*. The counterimage $T^{-1}\mathrm{SP}(B) \subseteq A$ is sometimes known [2],[18],[16] as the "generalized Fredholm" elements of A.

We remark that spectral permanence does not in general, or even together with one-one-ness, imply simple permanence: return to (3.8) and (1.5).

In general

$$\mathrm{SP}(A) \subseteq A^{\cup} \equiv \{a \in A : a \in aA^{-1}a\} \ , \tag{4.7}$$

and hence

$$\mathrm{SP}(A) \cap A^{-1}_{left} = A^{-1} = \mathrm{SP}(A) \cap A^{-1}_{right} \ . \tag{4.8}$$

This will show again that spectral permanence together with one one is not sufficient for generalized permanence:

Theorem 4.1. *If $B^{-1}_{left} \neq B^{-1}_{right}$ then there exist A and $T : A \to B$ for which T is one one with spectral but not generalized permanence.*

Proof. If A is commutative then $A^{\cap} = \mathrm{SP}(A)$ and hence

$$T(A^{\cap}) \subseteq \mathrm{SP}(B) \subseteq B^{\cap} \ ,$$

and if

$$T(A^{\cap}) \cap B^{-1}_{left} \setminus B^{-1} \neq \emptyset$$

then T does not have generalized permanence. Thus find $a \in B^{-1}_{left} \setminus B^{-1}$ and, recalling (1.9), take

$$T = J : \mathrm{comm}^2_B(a) \subseteq B$$

□

The familiar example is to take $B = L(X)$ to be the linear mappings on the space $X = \mathbf{C}^{\mathbf{N}}$ of all complex sequences and $a \in B$ to be the *forward shift*. Conversely however simple permanence together with one-one-ness does imply spectral permanence:

Theorem 4.2. *For semigroup homomorphisms*

$$\textit{one one and simple permanence implies spectral permanence} \ , \tag{4.9}$$

while conversely

$$\textit{simple and spectral permanence implies one one} \ . \tag{4.10}$$

Proof. The last implication is (3.8); conversely observe

$$\mathrm{SP}(A) \cap T^{-1}B^{-1}_{left} \subseteq A^{\cup} \cap T^{-1}B^{-1}_{left} \subseteq A^{-1} + T^{-1}(0) \tag{4.11}$$

□

When we specialise to rings of mappings then simple polarity is characterized by "ascent" and "descent":

Theorem 4.3. *If $A = L(X)$ is the additive, or linear, operators on an abelian group, or vector space, X then necessary and sufficient for $a \in A$ to be simply polar is that it has ascent ≤ 1,*

$$a^{-2}(0) \subseteq a^{-1}(0) \ ; \ \textit{equivalently } a^{-1}(0) \cap a(X) = O \equiv \{0\} \ , \tag{4.12}$$

and also descent ≤ 1,

$$a(X) \subseteq a^2(X) \ ; \ \textit{equivalently } a^{-1}(0) + a(X) = X \ . \tag{4.13}$$

The same characterization is valid when $A = B(X)$ for a Banach space X.

Proof. The complementary subspaces $a^{-1}(0)$ and $a(X)$ determine the idempotent $p : X \to X$, defined by setting

$$p(\xi) \in a(X) \ ; \ \xi - p(\xi) \in a^{-1}(0)$$

for each $\xi \in X$, whose boundedness, together with the closedness of the range $a(X)$, follows ([7] Theorem 4.8.2) from the open mapping theorem; and finally, if $\xi \in X$,

$$c(\xi) = cp(\xi) \ ; \ ca(\xi) = p(\xi)$$

□

We remark that, on incomplete spaces, the conditions (4.5) and (4.6) are not sufficient for simple polarity: indeed it is possible for $a \in B(X)$ to be one one and onto but not in $B(X)^\cap$: the obvious example is the "standard weight" $a = w$ on $X = c_{00} \subseteq c_0$ defined by setting

$$w(\xi)_n = (1/n)\xi_n \ .$$

Even together with the assumption $a \in A^\cap$, however, the conditions (4.5) and (4.6) are ([7] (7.3.6.8)) not sufficient for simple polarity (4.1) when $A = B(X)$ for an incomplete normed space X.

5. Drazin permanence

More generally if there is $n \in \mathbf{N}$ for which a^n is simply polar we shall also say that $a \in A$ is "polar", or *Drazin invertible.* If $a \in A$ is polar then there is $c \in A$ for which $ac = ca$ and $a - aca$ is *nilpotent.* If we further relax this to "quasinilpotent" we reach the condition that $a \in A$ "quasipolar". Specifically if we write

$$\mathrm{QN}(A) = \{a \in A : 1 - \mathbf{C}a \subseteq A^{-1}\} \tag{5.1}$$

for the *quasinilpotents* of a Banach algebra A then $a \in \mathrm{QN}(A)$ if and only if

$$\sigma_A(a) \subseteq \{0\} \ ,$$

while with some complex analysis we can prove that if $a \in \mathrm{QN}(A)$ then

$$\|a^n\|^{1/n} \to 0 \ (n \to \infty) \ . \tag{5.2}$$

In the ultimate generalization of "group invertibility", we shall write $\mathrm{QP}(A)$ for the *quasipolar* elements $a \in A$, those which have a *spectral projection* $q \in A$ for which (cf [8])

$$q = q^2 \ ; \ aq = qa \ ; \ a + q \in A^{-1} \ ; \ aq \in \mathrm{QN}(A) \ . \tag{5.3}$$

Now [17] the spectral projection and the *Koliha-Drazin inverse*

$$a^\bullet = q \ , \ a^\times = (a+q)^{-1}(1-q) \tag{5.4}$$

are uniquely determined and lie in the double commutant of $a \in A$. It is easy to see that if (5.3) is satisfied then

$$0 \notin \mathrm{acc}\ \sigma_A(a) \ \ : \tag{5.5}$$

the origin cannot be an accumulation point of the spectrum; conversely if (5.5) holds then we can display the spectral projection as a sort of "vector-valued winding number"

$$a^\bullet = \frac{1}{2\pi i}\oint_0 (z-a)^{-1} dz \ , \tag{5.6}$$

where we integrate counter clockwise round a small circle γ centre the origin whose connected hull $\eta\gamma$ is a disc whose intersection with the spectrum is at most the point $\{0\}$. Now generally for a homomorphism $T : A \to B$ there is inclusion

$$T\ \mathrm{QP}(A) \subseteq \mathrm{QP}(B) \ , \tag{5.7}$$

while if $T : A \to B$ has spectral permanence in the sense (1.3) then it is clear from (5.5) that there is also "Drazin permanence" in the sense that

$$\mathrm{QP}(A) = T^{-1}\mathrm{QP}(B) \subseteq A \ \ : \tag{5.8}$$

Theorem 5.1. *For Banach algebra homomorphisms $T : A \to B$ there is implication*

$$\textit{spectral permanence} \Longrightarrow \textit{Drazin permanence} \ .$$

Proof. Equality in (2.2), together with (5.5) $\square$

The example of Theorem 4.1 also shows that the left regular representation $L : A \to B(A)$, with $A = B(X)$ for a normed space X, does not always have generalized permanence; however we do have a sort of "closed range permanence": there is implication

$$L_a A = \mathrm{cl}\ L_a A \Longrightarrow a(X) = \mathrm{cl}\ a(X)\ : \tag{5.9}$$

indeed if $a\xi_n \to \eta$ and $\varphi \in X^*$ and $\varphi(\xi) = 1$ then, with $\varphi \odot \eta : \zeta \mapsto \varphi(\zeta)\eta$,

$$L_a(\varphi \odot \eta) = L_a(b) \Longrightarrow \eta = a(b\xi)\ . \tag{5.10}$$

Generally

Theorem 5.2. *If $T : A \to B$ is arbitrary then*

$$\mathrm{QP}(A) \cap T^{-1}(B^{-1}) \subseteq A^{-1} + T^{-1}(0) \tag{5.11}$$

and if $T : A \to B$ is one one then

$$\mathrm{QP}(A) \cap T^{-1}\mathrm{SP}(B) = \mathrm{SP}(A)\ . \tag{5.12}$$

Hence if $a \in B$ and $T = J : A = \mathrm{comm}^2_B(a) \subseteq B$ then

$$A^{\cap} = T^{-1}\mathrm{SP}(B)\ . \tag{5.13}$$

It follows that if $T^{-1}(0) = O$ then

Drazin $\Longrightarrow$ simple $\Longrightarrow$ spectral permanence .

Proof. Uniqueness guarantees that the spectral projection $T(a)^\bullet$ of $Ta \in \mathrm{SP}(B) \subseteq \mathrm{QP}(B)$ commutes with $T(a) \in B$, and one-one-ness guarantees the same for $a \in A$ $\square$

For Banach algebra homomorphisms therefore there is an improved version of Theorem 4.2: of the three conditions

spectral permanence ; simple permanence ; one one ,

any two imply the third.

If we rework Theorem 4.1 with $B = B(\ell_2)$ then it is clear that isometric homomorphisms with spectral permanence need not have generalized permanence: indeed the forward shift $a = u \in B^{\cap} \setminus \mathrm{QP}(A)$ is not even quasipolar: we recall that the spectrum of u is the closed unit disc, violating (5.5).

Theorem 4.1 was obtained in this way ([3] Theorem 3.2) in [3]. Of course (cf [9],[17]) "quasinilpotents" and"quasipolars" are only available in Banach algebras; Theorem 4.1 above, using "simply polar" elements, is conceptually much simpler.

6. Moore-Penrose permanence

We recall that a "C* algebra" is a Banach algebra which also has an *involution* $a \mapsto a^*$ which is conjugate linear, reverses multiplication, respects the identity and satisfies the "B* condition"

$$\|a^*a\| = \|a\|^2 \ (a \in A) \ . \tag{6.1}$$

Historically the term "C* algebra" was reserved for closed *-subalgebras of the algebras $B(X)$ for Hilbert spaces X; however the *Gelfand-Naimark-Segal* (GNS) representation

$$\Gamma : A \to B(\Xi_A) \tag{6.2}$$

takes an arbitrary "B* algebra" A isometrically into the algebra of operators on a rather large Hilbert space Ξ_A built from its "states": a defect of (6.2) would be that if already $A = B(X)$ we do not get back $\Xi_A = X$. In the opinion of this writer these terms "B* algebra" and "C* algebra" could easily ([7] Chapter 8) have been *Hilbert algebra*. When in particular $A = B(X)$ for a Hilbert space X then the closed range condition (3.9) is sufficient for relative regularity $a \in A^\cap$: indeed we can satisfy (2.2) by setting

$$c(\xi) = c(q\xi) \ ; \ c(a\xi) = p(\xi) \ (\xi \in X) \ , \tag{6.3}$$

where $q^* = q = q^2$ and $p^* = p = p^2$ are the orthogonal projections on the range $a(X)$ and the orthogonal complement $a^{-1}(0)^\perp$ of the null space. The element $c \in A$ given by (6.3) satisfies four conditions:

$$a = aca \ ; \ c = cac \ ; \ (ca)^* = ca \ ; \ (ac)^* = ac \ , \tag{6.4}$$

and is known as the *Moore-Penrose* inverse of $a \in B(X)$: more generally in a C* algebra A the conditions (6.4) uniquely determine at most one element

$$c = a^\dagger \in A \ , \tag{6.5}$$

lying ([11] Theorem 5) in the double commutant of $\{a, a^*\}$, and still known as a "Moore-Penrose inverse" for $a \in A$. Now it is a result of Harte and Mbekhta ([11] Theorem 6) that generally there is equality

$$A^\cap = A^\dagger \ : \tag{6.6}$$

in an arbitrary C* algebra, every relatively regular element has a Moore Penrose inverse. The argument, and a slight generalization, proceeds with the aid of the Drazin inverse.

More generally, on a semigroup A, an involution $a \mapsto a^*$ satisfies

$$(a^*)^* = a \ ; \ (ca)^* = a^*c^* \ ; \ 1^* = 1 \ . \tag{6.7}$$

In rings and algebras we also ask that the involution be additive, or conjugate linear. The B* condition (6.7) implies that, for arbitrary $a, x \in A$,

$$\|ax\|^2 \le \|x^*\| \; \|a^*ax\| \; , \tag{6.8}$$

which in turn gives *cancellation*

$$L^{-1}_{a^*a}(0) \subseteq L^{-1}_a(0) \; . \tag{6.9}$$

Generally the *hermitian* or "real" elements of A are given by

$$\mathrm{Re}(A) = \{a \in A : a^* = a\} \; . \tag{6.10}$$

The Moore-Penrose inverse $a^\dagger$ of (6.4), if it exists, is unique and double commutes with a and a^*. We pause to notice the *star polar* elements of a semigroup A:

$$\mathrm{SP}^*(A) = \{a \in A : a^*a \in A^\cap\} \; ; \tag{6.11}$$

now we claim

Theorem 6.1. *If the involution * on the semigroup A is cancellable then*

$$A^\dagger \subseteq \mathrm{SP}^*(A) \subseteq A^\cap \; . \tag{6.12}$$

Proof. With cancellation there is implication

$$a \in \mathrm{SP}^*(A) \Longrightarrow a \in aAa^*a \subseteq Aa^*a \cap aAa \; ,$$

and equality

$$\mathrm{Re}(A) \cap \mathrm{SP}^*(A) = \mathrm{Re}(A) \cap \mathrm{SP}(A) \; ,$$

If $a = aca$ with $c = a^\dagger$ then

$$a^*a = a^*(ac)(ac)^*a = a^*acc^*a^*a \in a^*aAa^*a \;\; :$$

conversely, by cancellation,

$$a^*a = ada^*a \Longrightarrow a = ada^*a \;\; :$$

hence also

$$a \in Aa^*a \; ; \; \Longleftrightarrow a^* \in a^*aA \; .$$

Hence if $a^* = a$ then (4.2) follows □

It is now clear that an isometric C* homomorphism has "Moore-Penrose permanence":

Theorem 6.2. *If $T : A \to B$ has simple permanence then*

$$T^{-1}B^\dagger \subseteq A^\dagger \; . \tag{6.13}$$

Proof. We claim

$$A^\dagger = \{a \in A : a^*a \in \mathrm{SP}(A)\}\ , \tag{6.14}$$

with implication

$$a^*a \in \mathrm{SP}(A) \Longrightarrow a^\dagger = (a^*a)^\times a^*\ .$$

If $a \in A^\dagger$ with $a = aca$ and $(ca)^* = ca$ then, with $d = cc^*$, we have

$$a^*ad = a^*acc^* = a^*c^* = a^*c^*a^*c^* = ca$$

and

$$da^*a = cc^*a^*a = ca\ .$$

Conversely if $a^*a = a^*ada^*a$ with $a^*ad = da^*a$ with (wlog) $d = d^*$ then, using cancellation, with $c = da^*$,

$$aca = ada^*a = a \text{ and } ca = da^*a = a^*ad = a^*c^*\ .$$

Now if $a \in A$ there is implication

$$Ta \in B^\dagger \Longrightarrow T(a^*a) \in \mathrm{SP}(B) \Longrightarrow a^*a \in \mathrm{SP}(A) \Longrightarrow a \in A^\dagger$$

□

Our main result is a slight generalization, and a new proof, of the Harte/Mbekhta result (6.6), and at the same time "generalized permanence", equality in (3.4), for isometric C* homomorphisms. One way to go, thanks to the Gelfand/Naimark/Segal representation, is to look first in the very special algebra $D = B(X)$ of bounded Hilbert space operators:

Theorem 6.3. *If* $d \in D = B(X)$ *for a Hilbert space* X *then*

$$(d^*d)^{-1}(0) \subseteq d^{-1}(0) \tag{6.15}$$

and

$$\mathrm{cl}\ d(X) + d^{*-1}(0) = X\ ; \tag{6.16}$$

hence if cl $d(X) = d(X)$ *then*

$$d^*(X) = d^*d(X)\ , \text{andcl}\ d^*d(X) = d^*d(X)\ . \tag{6.17}$$

There is inclusion

$$\mathrm{Re}(D) \cap D^\cap \subseteq \mathrm{SP}(D)\ ; \tag{6.18}$$

hence

$$d \in D^\cap \Longrightarrow d \in \mathrm{SP}^*(D) \Longrightarrow d^*d \in \mathrm{SP}(D) \Longrightarrow d \in D^\dagger\ . \tag{6.19}$$

Proof. For arbitrary $\xi \in X$ there is [3] inequality

$$\|d\xi\|^2 \leq \|\xi\| \ \|d^* d\xi\| \ ,$$

and also

$$\mathrm{cl}\ d(X) = d^{*-1}(0)^\perp$$

□

Both of the Harte/Mbekhta observations now follow:

Theorem 6.4. *If $T : A \to B$ is isometric then*

$$T^{-1}(B^\cap) \subseteq A^\dagger \ . \tag{6.20}$$

Proof. With $S : B \to D = B(X)$ a GNS mapping we argue, using again Theorem 4.2, together with "spectral permanence at" $a^* a$ (which has of course real spectrum),

$$Ta \in B^\cap \Longrightarrow ST(a^* a) \in \mathrm{SP}(D) \Longrightarrow a^* a \in \mathrm{SP}(A) \Longrightarrow a \in A^\dagger$$

□

In the situation of (6.14),

$$a = a^* \in A^\cap \Longrightarrow a^\dagger = a^\times \ ; \ 1 - a^\dagger a = a^\bullet \ . \tag{6.21}$$

Theorem 6.4 has an obvious extension to homomorphisms with closed range:

Theorem 6.5. *If $T : A \to B$ has closed range then there is implication, for arbitrary $a \in A$,*

$$T(a) \in B^\cap \Longrightarrow a + T^{-1}(0) \in (A/T^{-1}(0))^\cap \ . \tag{6.22}$$

Proof. Apply Theorem 6.4 to the bounded below $T^\wedge : A/T^{-1}(0) \to B$ □

References

[1] SR Caradus, *Generalized inverses and operator theory*, Queens papers Pure Appl. Math. Ontario **50** (1978)

[2] SR Caradus, WE Pfaffenberg and B Yood, *Calkin algebras and algebras of operators on Banach spaces*, Dekker 1974.

[3] DS Djordjević, S Zivković-Zlatanović and RE Harte, *Spectral permanence and the Moore-Penrose inverse*, Proc. Amer. Math. Soc. **140** (2012) 8237-45.

[4] DS Djordjević, JJ Koliha and I Straskraba, *Factorization of EP elements of C* algebras*, Linear Multilinear Alg. **57** (2009) 587-594.

[5] DS Djordjević and V Rakocević, *Lectures on generalized inverses*, University of Nis 2008.

[6] JJ Grobler and H Raubenheimer, *Spectral properties of elements of different Banach algebras*, Glasgow Math. Jour. **33** (1991) 11-20.

[7] RE Harte, *Invertibility and singularity*, Dekker 1988.
[8] RE Harte *Spectral projections*, Irish Math. Soc. Newsletter **14** (1984) 10-15.
[9] RE Harte, *On quasinilpotents in rings*, PanAmerican Jour. Math. **1** (1991) 10-16.
[10] RE Harte, *Unspectral sets*, Rendiconti Mat. **56** (1998) 69-77.
[11] RE Harte and M Mbekhta, *On generalized inverses in C* algebras*, Studia Math. **103** (1992) 71-77.
[12] RE Harte and M Mbekhta, *On generalized inverses in C* algebras II*, Studia Math **106** (1993) 129-138.
[13] RE Harte and M O'Searcoid, *Positive elements and the B* condition*, Math. Zeit. **193** (1986) 1-9.
[14] RE Harte and AW Wickstead, *Boundaries, hulls and spectral mapping theorems*, Proc. Royal Irish Acad. **81A** (1981) 201-208.
[15] R Hartwig and J Lu, *On the group structure of unit regular ring elements*, Pacific Jour. Math. **71** (1977).
[16] G Kantun-Montiel, SV Djordjević and RE Harte, *On semigroup inverses*, Funct. Anal. Approx. Comp. **1** (2009) 11-18.
[17] JJ Koliha, *A generalized Drazin inverse*, Glasgow Math. Jour. **38** (1996) 367-381.
[18] Ch Schmoeger, *On a class of generalized Fredholm operators* I-VII, Demonstratio Math. **30-33** (1997-2000).
[19] M Xavier, *On the converse of a theorem of Harte and Mbekhta*, Studia Math. **184** (2008) 149-152.
[20] M Xavier, *On generalized inverses and Green's relations*, Linear Alg. Appl. **434** (2011) 1830-1844.

Robin Harte taught for twenty years at University College, Cork, and retired a long time ago, but somehow never goes away: his ghost can sometimes be seen at the TCD-UCD Analysis Seminar.

School of Mathematics, Trinity College Dublin
E-mail address: rharte@maths.tcd.ie

Irish Math. Soc. Bulletin
Number 69, Summer 2012, 47–56
ISSN 0791-5578

THE EQUATION OF TIME AND THE ANALEMMA

PETER LYNCH

Abstract. The Earth's progress around the Sun varies through the year. Combined with the tilt of the axis of rotation, this results in variations of the length of a solar day. The variations are encapsulated in the Equation of Time. A plot of altitude versus azimuth for the Sun at 12 noon local time through the year describes a figure-of-eight curve known as an analemma. By analysis of the observations, we find that the qualitative aspects of the analemma can be reproduced using just two sinusoidal components.

1. Introduction

An analemma is the curve obtained by plotting the position of the Sun, as viewed from a fixed location on Earth, at the same clock time each day for a year. If the Earth's orbit were perfectly circular and the axis of rotation were perpendicular to the plane of the orbit, the analemma would collapse to a fixed point. However, the orbit is elliptical and the axis tilted, and the analemma is a large figure-of-eight. This has important consequences for the measurement of time.

On the East Pier in Dun Laoghaire there is an analemmatic sundial. The hour-points are on an ellipse, the horizontal projection of a circle parallel to the equator. The gnomon is formed by the observer, whose shadow falls on the ellipse, indicating the time. Three adjustments must be made to get mean time from sun-dial time. First, since Dun Laoghaire is six degrees, eight minutes west of Greenwich, 25 minutes must be added. Next, a seasonal correction must be made. This is read from a graph of the Equation of Time, conveniently plotted on a bronze plaque (Fig. 1). Finally, an extra hour must be added during Irish Summer Time. In this

2010 *Mathematics Subject Classification.* 85-01, 85A04.
Key words and phrases. Mean Time, Analemma.
Received on 10-8-2012; revised 24-8-12.

©2012 Irish Mathematical Society

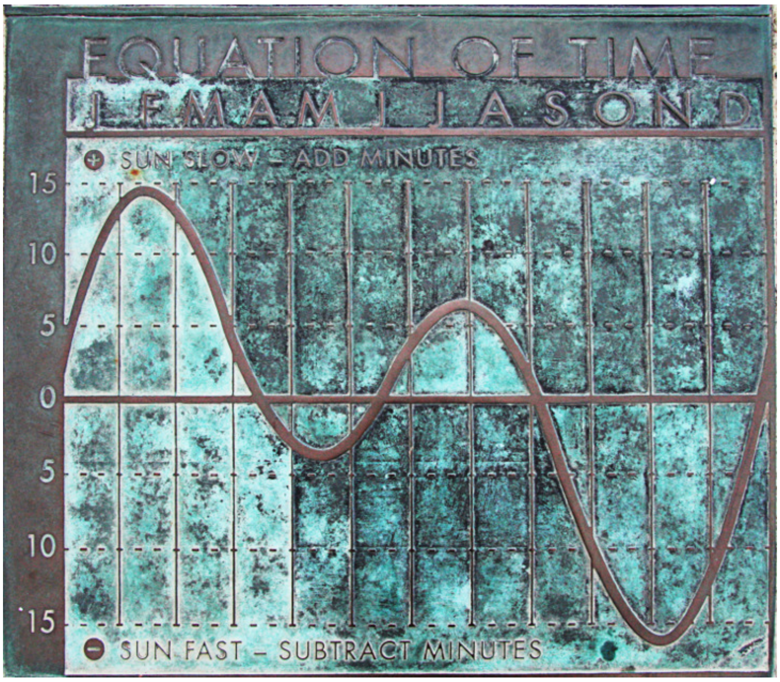

FIGURE 1. Bronze plaque indicating the Equation of Time, on the analemmatic sundial installed on the East Pier in Dun Laoghaire (Graph drawn by Capt. Owen M. Deignan. *Photo*: Peter Lynch).

paper, we will examine the Equation of Time and show how it can be expressed approximately in terms of two sinusoidal components.

2. Observations

The position of the Sun in the sky, as seen from the Royal Observatory, Greenwich at 12:00 GMT each day for 2006, is available online [5]. The position is specified by two angles, analogous to latitude and longitude: the *altitude*, or angle relative to the horizon and the *azimuth*, or angle relative to true north. A plot of altitude versus azimuth (Fig. 2) describes a figure-of-eight curve known as an analemma.

The apparent variation in the position of the Sun has been intensively studied by astronomers, and is well understood. It is the cause of variations in the length of a solar day and the difference between solar time and mean time. The variations are encapsulated in an expression called the Equation of Time (the term ‘equation’ is used here in a historical sense, meaning a correction or adjustment). The difference between mean and solar time can be predicted with

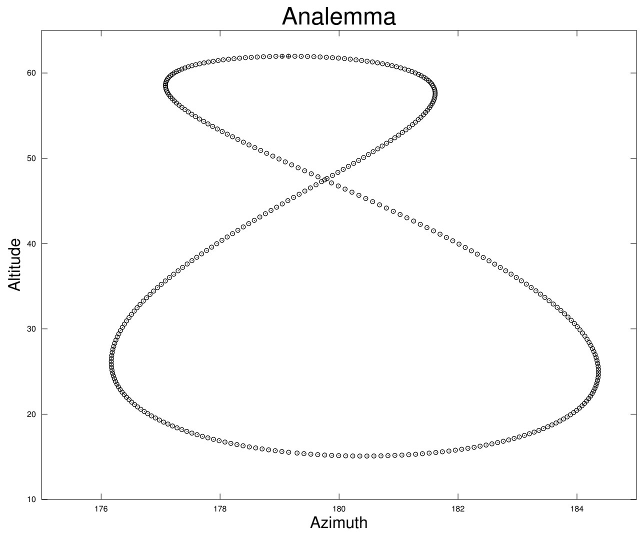

FIGURE 2. The analemma, based on observed values of the azimuth and altitude of the Sun at 12.00 GMT at the Royal Observatory, Greenwich for 2006 (see [5]). Note the unequal axes.

great precision. For example, [2] gives an algorithm that calculates the Equation of Time valid over a period of 6000 years, accurate to within three seconds. So, accurate estimation of the correction is not an issue. However, examination of the data shows that there are dominant components of the Equation of Time that call for explanation, and it is instructive and illuminating to examine these and explain them in terms of the main variations in the Earth's orbit. This is the goal of the present note.

A Fourier analysis of the altitude and azimuth for the year 2006 shows that only the first few components have appreciable amplitude. Fig. 3 (left panel) makes it clear that the altitude is dominated by the component with a period of a year; the Sun moves between the Tropics of Cancer and Capricorn in an essentially sinusoidal fashion. In Fig. 3 (right panel), the amplitudes of the coefficients of the transformed azimuth show that components 1 and 2 are dominant; component 3 is not negligible, but it is substantially smaller than the two main components. Thus, the principal variations in azimuth have periods of a year and a half year.

When the altitude is plotted against the azimuth, the figure-of-eight curve shown in Fig. 2 results. The pattern can be further broken down by taking the in-phase and quadrature elements of each

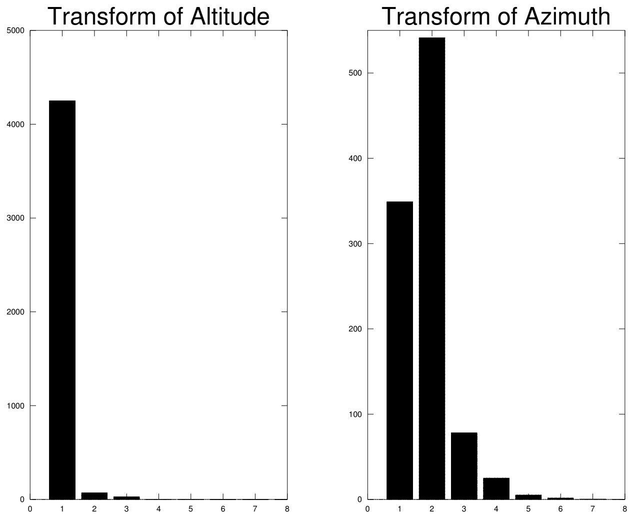

FIGURE 3. Magnitude of the Fourier components of altitude (left) and azimuth (right). Only the first 8 components are shown.

of the two components of the azimuth. Thus, for the annual component, we plot the part in phase with the altitude in Fig. 4(A) and the part orthogonal to altitude in Fig. 4(B). Similarly, the two parts of the semi-annual component of azimuth are plotted in Fig. 4(C) and Fig. 4(D).

The aim of the remainder of this paper is to explain the observed variations in terms of the characteristics of the orbit of the Earth. There are two main variations, the eccentricity of the Earth's elliptical orbit and the obliquity, or tilt of the axis relative to the ecliptic or plane of the orbit around the Sun. We will examine each in turn. We remark that highly accurate values for all the quantities and expressions that we consider are available in the astronomical literature. The present note is concerned with elucidating mechanisms rather than with precision.

3. VARIATIONS DUE TO ELLIPTICITY OF THE ORBIT

To a high degree of approximation, the Earth's orbit is a Keplerian ellipse. Perihelion in 2006 was on 4 January and aphelion on 3 July. The Earth rotates about its axis in a sidereal day and revolves about the Sun in a year, so the ratio of mean angular velocity of revolution ϖ to that of rotation Ω is about 1/365.

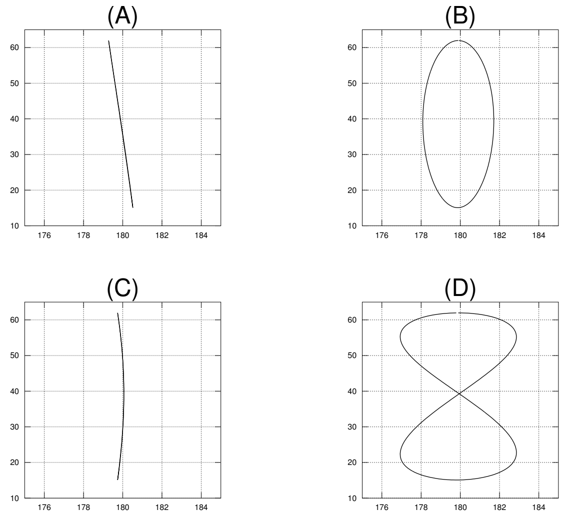

FIGURE 4. Lissajous components of the analemma. A: In-phase annual component; B: Quadrature annual component; C: In-phase semi-annual component; D: Quadrature semi-annual component.

By Kepler's Second Law, the angular momentum (per unit mass) $h = r^2\dot{\theta}$ is constant [6]. Here, r is the distance between the Earth and Sun and θ is the 'true anomaly', the angle between the radius vector and the line from the Sun to the perihelion. Let a be the semi-major axis and e the eccentricity of the orbit. The perihelion and aphelion distances are respectively $r_{\mathrm{P}} = (1-e)a$ and $r_{\mathrm{A}} = (1+e)a$ and, if the angular velocities at these points are $\omega_{\mathrm{P}} = \dot{\theta}_{\mathrm{P}}$ and $\omega_{\mathrm{A}} = \dot{\theta}_{\mathrm{A}}$, we have

$$h = (1-e)^2 a^2 \omega_{\mathrm{P}} = (1+e)^2 a^2 \omega_{\mathrm{A}} \,.$$

Since the eccentricity is small ($e \approx 0.0167$), we can take the mean angular velocity to be $\varpi = h/a^2$. Then

$$\omega_{\mathrm{P}} = h/[(1-e)a]^2 \approx (1+2e)\varpi$$
$$\omega_{\mathrm{A}} = h/[(1+e)a]^2 \approx (1-2e)\varpi$$

As the Earth rotates and revolves, the Sun appears to revolve about it with angular velocity $\Omega - \omega$, so the difference between the rates at aphelion and perihelion is

$$(\Omega - \omega_{\mathrm{A}}) - (\Omega - \omega_{\mathrm{P}}) = (\omega_{\mathrm{P}} - \omega_{\mathrm{A}}) \approx 4e\varpi$$

Thus, the fractional change is $4e\varpi/\Omega$. This determines the length of a solar day, which may be shorter or longer than the mean day by an amount $2e\varpi/\Omega \approx 9.15 \times 10^{-5}\,\mathrm{s\,s^{-1}}$, or 7.9 seconds in a day. This is small, but it accumulates over a period of weeks or months.

The variation in the length of a solar day can be approximated by a sinusoidal wave with amplitude $2e\varpi/\Omega$, varying on an annual cycle:

$$\Delta_{\mathrm{ECC}} = \frac{2e\varpi}{\Omega}\cos M_1 \tag{1}$$

where $M_1 = 2\pi(D - D_{\mathrm{P}})/365$ with D the day number and D_{P} the date of perihelion. Eccentricity causes a lengthening of the solar day at perihelion (near mid-winter) and a shortening at aphelion (near mid-summer), and Δ_{ECC} is the amount that must be added to correct solar time for the effect of the Earth's elliptic orbit.

4. Variations due to obliquity of the orbit

Let us now disregard the eccentricity temporarily, and assume that the Earth's orbit is circular. If the axis of rotation were perpendicular to the ecliptic, or plane of the Earth's orbit around the Sun, each day would be the same length. The Earth would advance by about 1° during the course of a sidereal day, so a solar day would be longer by a factor of about $\frac{1}{360}$, or about 4 minutes. However, the equatorial plane of Earth is tilted to the ecliptic by an angle $\epsilon \approx 23.44^\circ$, called the *obliquity*. So, the ecliptic plane cuts the earth in a great circle that makes an angle ϵ with the equator at the two points where they intersect. These points correspond to the equinoxes.

Let us also disregard the Earth's rotation momentarily; effectively, we are taking a stroboscopic view with a frequency Ω. Then, during the course of a year, the Sun will trace out the great circle at a constant rate. The equation for the great circle is an elementary geometrical exercise; the latitude (ϕ) and longitude (λ) are related by

$$\tan\phi = \tan\epsilon\sin(\lambda - \lambda_0) \tag{2}$$

where λ_0 corresponds to the vernal equinox. For simplicity, we set $\lambda_0 = 0$. Eqn. (2) corresponds to one of Napier's rules for right spherical triangles; see [8, p. 888]. Differentiating (2), we get

$$\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = \frac{\tan\epsilon\cos\lambda}{1 + \tan^2\epsilon\sin^2\lambda}\,.$$

The progress of the Sun along the trajectory (2) is constant, but the change in longitude, which determines the time, is not. If we consider a small step $\mathrm{d}\sigma$ along the great circle, the spherical metric gives $\mathrm{d}\sigma^2 = \cos^2\phi\,\mathrm{d}\lambda^2 + \mathrm{d}\phi^2$, so the change in λ with respect to σ is

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\sigma} = (1 + \tan^2\epsilon\sin^2\lambda)\cos\epsilon\,.$$

The extreme values follow immediately. For small or moderate values of the obliquity ϵ, they are:

$$\begin{array}{ll} \text{Equinoxes } (\lambda = 0, \pi): & \dfrac{\mathrm{d}\lambda}{\mathrm{d}\sigma} = \cos\epsilon \approx 1 - \tfrac{1}{2}\epsilon^2 \\ \text{Solstices } (\lambda = \dfrac{\pi}{2}, \dfrac{3\pi}{2}): & \dfrac{\mathrm{d}\lambda}{\mathrm{d}\sigma} = \sec\epsilon \approx 1 + \tfrac{1}{2}\epsilon^2 \end{array}$$

Thus, the solar day may be shorter or longer than the mean day by an amount $\frac{1}{2}\epsilon^2\varpi/\Omega \approx 2.29 \times 10^{-4}\,\mathrm{s\,s^{-1}}$, or 19.8 seconds in a day. We note that [3] gives a more accurate expression where the factor $\frac{1}{2}\epsilon^2$ is replaced by $2\tan^2\frac{\epsilon}{2}$. However, our concern here is less with precision and more with simplicity.

The variation in the length of a solar day can be approximated by a sinusoidal wave with amplitude $\frac{1}{2}\epsilon^2\varpi/\Omega$, varying on a semi-annual cycle:

$$\Delta_{\mathrm{OBL}} = \frac{\epsilon^2\varpi}{2\Omega}\cos 2M_2 \tag{3}$$

where $M_2 = 2\pi(D - D_{\mathrm{W}})/365$ with D the day number and D_{W} the date of the winter solstice (day 355 in 2006). Obliquity causes a lengthening of the solar day at the equinoxes and a shortening at the solstices, and Δ_{OBL} is the amount that must be added to correct solar time for its effect.

5. The Equation of Time

We now combine the effects of eccentricity (1) and obliquity (3) to get the total difference between solar and mean time, $\Delta = \Delta_{\mathrm{ECC}} + \Delta_{\mathrm{OBL}}$. To calculate the accumulated difference, this must be integrated, to give

$$\begin{aligned} E &= \frac{2e\varpi}{\Omega}\sin M_1 + \frac{\epsilon^2\varpi}{4\Omega}\sin 2M_2 \\ &= 7.9\sin\left(\frac{2\pi(D-3)}{365}\right) + 9.9\sin 2\left(\frac{2\pi(D-355)}{365}\right)\,. \end{aligned} \tag{4}$$

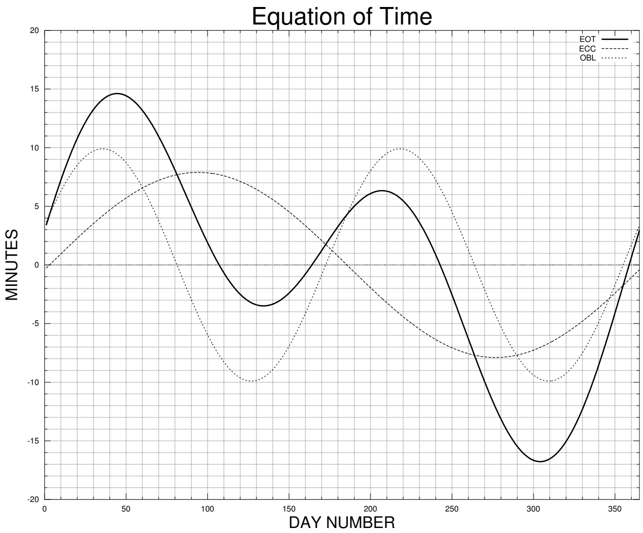

Figure 5. Equation of Time (in minutes) as a function of the day number, computed using (4). The components due to eccentricity (dashed) and obliquity (dotted) are shown. This is the correction that must be added to solar time to get mean time.

This gives the difference between the mean and solar time in minutes. It varies by about 15 minutes in both directions. The approximate curve, together with the two components, due to eccentricity and obliquity, are shown in Fig. 5. There is good qualitative agreement with the curve in Fig. 1.

To construct an approximation to the analemma, we convert time in minutes given by (4) to degrees longitude by dividing by 4 and adding 180°. The approximate and observed curves are plotted in Fig. 6. We see that the main features of the observed pattern are replicated, but there are significant differences. These discrepancies can be reduced by including higher terms. A very precise, but more complicated, description of the Equation of Time is given in [2].

6. Discussion

The difference between mean time and solar time is expressed as the Equation of Time. Fourier analysis of the observations at the Royal Observatory in Greenwich shows that the variations in the Sun's noontime position are dominated by the first few Fourier coefficients. This allows us to approximate the Equation of Time by two

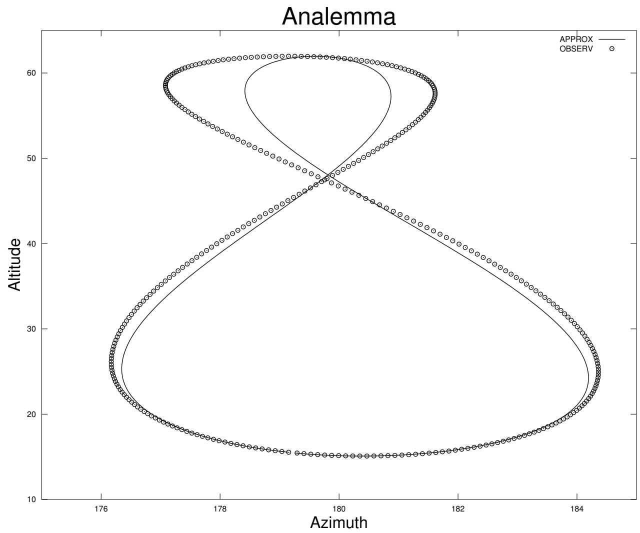

Figure 6. Solid line: analemma based on the approximate Equation of Time (4). Dotted line: analemma based on observations, as in Fig. 2.

sinusoidal components, with periods of a year and a half year. The curve that results from plotting the resulting approximation against solar altitude is qualitatively similar to the observed analemma.

The analemma has many applications. It can be used to estimate the time and azimuth of sunrise and sunset, and to explain the occurrence of latest sunset some days before the winter solstice and the earliest sunrise some days later. Many geosynchronous, but not geostationary, satellites move on analemmatic curves, and the control systems for the paraboloidal dishes used to track them must compute these curves to ensure optimum communications. Finally, the Equation of Time is important in many scientific and engineering contexts. It is used for the design of solar trackers and heliostats, vital for harnessing solar energy.

If more precise approximations of the Equation of Time are required, they are available in [2]. Practical information on the construction of analemmatic sundials is presented in [1] and [7], and a program to compute the design for a given location is given in [4].

References

[1] Budd, C. J. and C. J. Sangwin, 2001: *Mathematics Galore.* Oxford Univ. Press., 254pp.

[2] Hughes, D. W., B. D. Yallop and C. Y. Hohenkerk, 1989: The equation of time. *Mon. Not. Roy. Astr. Soc.*, **238**, 1529–1535.

[3] Milne, R. M., 1921: Note on the Equation of Time. *Math. Gaz.*, **10**, 372–375.

[4] Pruss, Alexander R., 2011: Analemmatic Sundial PDF Generator (program to design an analemmatic sundial, given the latitude, longitude and time-zone). `http://analemmatic.sourceforge.net/cgi-bin/sundial.pl`

[5] RGO, 2006: Azimuth and altitude of Sun at Royal Observatory, Greenwich, 2006. Available online at `http://en.wikipedia.org/wiki/File:Analemma_Earth.png`

[6] Roy, A. E., 2005: *Orbital Motion.* 4th Edn. Inst. of Phys. Publ. 592pp.

[7] Sangwin, Chris and Chris Budd, 2000: Analemmatic sundials: how to build one and why they work. *Plus Magazine*, Issue 11, June 2000. `http://plus.maths.org/content/os/issue11/features/sundials/index`

[8] Vallado, D. A., 2001: *Fundamentals of Astrodynamics and Applications.* Published jointly by Microcosm Press, Calif. and Kluwer Acad. Press Dordrecht, 958pp.

Peter Lynch is Professor of Meteorology at UCD. His interests include dynamic meteorology, numerical weather prediction, Hamiltonian mechanics and the history of meteorology. He writes an occasional mathematics column *That's Maths* in the *Irish Times*; see his blog at `http://thatsmaths.com`.

School of Mathematical Sciences, University College Dublin
E-mail address: `Peter.Lynch@ucd.ie`

Irish Math. Soc. Bulletin
Number 69, Summer 2012, 57–59
ISSN 0791-5578

Seán Dineen: Analysis – A Gateway to Understanding Mathematics, World Scientific, 2012, ISBN:978-9814401388

REVIEWED BY TOM CARROLL

The book under review is a first course not only in analysis and calculus but in the culture of mathematics. It grew from lecture notes for a mathematics course aimed at Economics and Finance students at University College Dublin and caters not only for mathematics students but for students whose area of primary interest lies outside mathematics. It is clear that the author believes that all students, including those who would typically be classed by a mathematics department as taking a 'service course', should be expected to understand the principles of mathematics and be skilled in their use.

Dineen paints on a broad canvas. The topics standard to all calculus and analysis textbooks are covered completely and in detail – number, function, limits and continuity, sequences and series, differentiation, integration, applications – with the approach being thorough right from the beginning. The reader is encouraged to think about each topic from different points of view. Rather than assuming the role of the omniscient author who providentially introduces material in advance of needing it, Dineen deals with issues only as they arise and introduces new mathematics only as it is needed. For example, though proofs are centre stage throughout the book, readers pick up proof techniques by degrees and in a manner commensurate with their growing mathematical maturity and confidence. That this approach works is partly due to the historical narrative which runs parallel to the mathematics and which explains how central concepts, such as number, function, limit, came into focus only gradually and were used implicitly long before their modern definitions solidified, often long before it was even realized that a precise definition might be needed. Pen pictures of the main contributors to the development of mathematics give the narrative a human feel and reinforce the message that understanding this mathematics takes

Received on 7-8-2012.

©2012 Irish Mathematical Society

time and commitment. Teachers' expectation of engagement on the part of students needs to be matched by a reciprocal commitment to engage with students and to bring students along with them. Dineen does this through a non-linear mathematical narrative. In the first chapter on Quadratic Functions, the rules of algebra, cancelation, square roots, functions are used freely, even if they will be formally introduced only later on. Though the exponential function and the logarithm function (modulo the intermediate value theorem which is proved later) are first formally defined in Chapter 7, these functions are introduced informally in Chapter 2 and are used freely throughout the book. Real Numbers feature right from the start – that positive numbers have a square root is used in Chapter 1, the completeness axiom is the basis of Chapter 6 where all the main properties of supremum and infimum of sets of real numbers are proved – even if the completeness axiom and the construction of the real numbers comes later in Chapter 12. It is in this sense that the narrative is 'non-linear'. It has taken great care and thought on the part of the author to ensure that this approach works logically, which it most certainly does. Dineen is frank about his approach. He introduces concepts gradually, informally at first, with an emphasis on understanding rather than absolute rigour. He aims to 'blend intuitive techniques and rigourous definitions' with the rigor coming later, often motivated by the historical realization that clarity and precision would be essential if further progress was to be made (cf, for example, Berkeley's criticisms of Newton and Leibniz's calculus, which Dineen discusses in detail). Dineen's approach steers a careful course between the ubiquitous calculus tome and a potentially dry first course in mathematical analysis and, in so doing, is more reflective of the way we learn mathematics. Let me be entirely clear that, though the mathematical development often moves ahead of itself only to regroup later, the narrative is entirely consistent and leaves no loose ends.

To conclude, a technical word or two about the author's mathematical choices. The rational numbers are constructed from the positive rational numbers which are in turn constructed directly from the natural numbers, so that the integers come after the rationals in Dineen's development. This works very well and is, of course, a perfect opportunity to bring in equivalence relations. Countability is covered in the context of number and function. Analysis is based on sequences, which avoids any ϵ-δ arguments. In fact, Dineen avoids ϵ entirely by restricting himself first to monotonic sequences, defining the limit of a bounded monotonic sequence to be the supremum if it is increasing, or the infimum if it is decreasing, of the set of real numbers which occur in the sequence. Having consolidated this notion through a variety of examples and results, Dineen defines a general sequence to be convergent if it lies between an increasing and a decreasing sequence which are convergent and have the same limit. All aspects of infinite series are covered in detail. From here he naturally defines continuity of a function at a point to be sequential continuity. I particularly enjoyed Chapter 10 on the construction of the real numbers using sections of the dyadic rationals and a 'bisection principle'. The last sections of the book cover first the derivative and its applications, then the Riemann integral for continuous functions and its applications. A notable feature of the book are the numerous well thought out, interesting exercises at the end of each chapter, with solutions provided at the end of the book.

Tom Carroll is a senior lecturer in mathematics at University College Cork. His main areas of mathematical interest are complex analysis and potential theory

School of Mathematical Sciences, University College, Cork
E-mail address: t.carroll@ucc.ie

Irish Math. Soc. Bulletin
Number 69, Summer 2012, 60–62
ISSN 0791-5578

Persi Diaconis and Ron Graham: Magical Mathematics, Princeton University Press, 2012, ISBN:978-0-691-15164-9

REVIEWED BY FIACRE Ó CAIRBRE

The authors have produced an invaluable contribution to the fascinating relationship between magic and mathematics. There is an engaging account of the history and characters behind the pieces of magic. The authors delve into some non–trivial mathematics behind the tricks and also give other applications of the relevant mathematics. This attractively produced book provides an enlightening insight into the crossover between mathematics and magic. Step by step instructions are given for each trick and so after reading it you may be able to impress your family and friends with some stunning magic. Both authors are professional mathematicians and one is a professional magician while the other is a professional juggler.

The authors exhibit a lifelong passion, enthusiasm and deep knowledge for magic and mathematics and this is an ideal combination for producing a great read. As a thirteen year old, one of the authors was a regular at the world famous Tannen's Magic Emporium in New York's Times Square. The shop was like a wonderland for a young boy interested in magic. An endearing account of some of the other regulars at Tannen's is provided. For example, one regular is described as follows: "There's Manny Kraut, a huge man whose fat hands somehow make the most beautiful, delicate card tricks".

One of the earliest tricks discussed goes back to Fibonacci's famous book, Liber Abaci, which appeared in 1202. The first serious magic books (two of them coincidentally in the same year) appeared in 1584. The current state of magic today is very vibrant and is described as "a very active whirlpool". The authors say "there are a handful of inventors who are repeatedly brilliant"and seven of them are discussed in the book. It's quite a diverse lot. There is a chicken farmer from Petaluma, a rural free-delivery mailman, a computer

Received on 11-5-2012.

©2012 Irish Mathematical Society

wizard (or two) and a priest. One was a hobo who lived out of dumpsters.

The book contains a wide diversity of types of tricks including cards, coins, pictures, words, paper folding and chains. My three favourite tricks involve de Bruijn sequences, the Gilbreath Principle and the so called Miracle Divination. A de Bruijn sequence with window length k is a zero/one sequence of length 2^k such that every k consecutive digits appear just once (going around the corner). For example, 11100010 is a de Bruijn sequence of window length 3. If one has a de Bruijn sequence of window length k, then one can perform the trick with 2^k cards. This is a great trick that should impress any audience. Continuing in the tradition of the magician keeping the details of the trick secret from the audience, I won't give away the details of the trick. I think it's better that one reads the book in order to see the details of the tricks. Some non–card trick applications of de Bruijn sequences and higher dimensional de Bruijn arrays are discussed including robotic vision, industrial cryptography, DNA, protein folding and rhyming patterns in East Indian music. The authors mention some open problems in mathematics concerning higher dimensional de Bruijn arrays. They also give some details about a recently solved problem related to Hamiltonian cycles.

The Gilbreath Principle is named after Norman Gilbreath who was an undergraduate in mathematics when he created an ingenious new card trick that would stun the world of magic. In July 1958, Gilbreath introduced himself in the magic magazine, Linking Ring, as follows: "I have been interested in magic for ten years. I am a math major at UCLA. Being a supporter of the art of magic, I have created over 150 good tricks and many others not so good. Here are a couple I hope you can use". He then gave a brief account of what is now called Gilbreath's First Principle. His new card trick was picked up and varied almost immediately by the magic community. In the January 1959 issue of Linking Ring, card experts Charles Hudson and Edward Marlo wrote "It is not often one runs across a new principle in card magic ... Gilbreath's principle has proved the most popular card effect to appear in the parade for a long time". In a 1966 issue of Linking Ring, Gilbreath, who was now a professional mathematician working for the Rand Corporation, introduced his so called Second Principle, which included new uses for his First Principle and many non–card tricks.

There is a surprising connection between Gilbreath's Principle and the Mandelbrot set. The authors were informed about this connection by Dennis Sullivan who attributes its proof to the Fields Medallists, John Milnor and William Thurston. This story illustrates that the mathematics behind card tricks may be a lot deeper than one might think. Some other non–magic applications of Gilbreath's Principle are given, including applications to Penrose tiles and designing sorting algorithms for computers.

The Miracle Divination is a coin trick involving three spectators. It has a long history with one version going as far back as Fibonacci's 1202 book, Liber Abaci.

The authors perform their magic to an eclectic audience. For example, in relation to the trick involving de Bruijn sequences above they say: "The trick is one we have performed for drunks in seedy nightclubs, at Hubert's Flea museum and at the banquet for the American Mathematical Society".

This illuminating book was a pleasure to read and I highly recommend it to anybody interested in the mathematics behind some impressive magic.

Fiacre Ó Cairbre is a Senior Lecturer in the Department of Mathematics and Statistics at NUI, Maynooth. His research interests are currently in the three areas of stability theory, mathematics education and the history of mathematics.

Department of Mathematics and Statistics, NUI, Maynooth
E-mail address: fiacre.ocairbre@nuim.ie

Irish Math. Soc. Bulletin
Number 69, Summer/Winter 2012, 63–66
ISSN 0791-5578

Dana Mackenzie: The Universe in Zero Words, Princeton University Press, 2012, ISBN:978-0-691-15282-0

REVIEWED BY ANTHONY G. O'FARRELL

This is a 'concept' book. Elwin Street Productions describe themselves as follows:

> We're a lively, independent illustrated coedition publisher. We conceive and produce a mix of stylish reference, handbooks and giftbooks that combine superlative writing and strong concepts with an off-beat sensibility and a fresh, spirited feel.

Their concepts have resulted in a number of series such as *The Little Book of x, x in Your Pocket, How to be an x, The curious Girl's Book of x, Freaky x, x for Busy People* (for x here and later substitute a topic such as Climate Change, Conspiracies, Romans, Outer Space, Campfire Cooking, etc.), and many more individual ideas. They conceived the idea of a book about equations, and went looking for a writer. They found Dana Mackenzie, who had established himself as a popular science writer after 13 years as an academic mathematician. He wrote the book, and they placed it with PUP, and so here we are.

There is a recognised need for mathematicians to communicate with the wider public, and there is an appetite out there for digestible material, so although the foregoing scenario does not resemble our usual model (in which we think up the whole idea from the start), I guess it makes sense.

MacKenzie has the knack of getting and keeping your attention, and writes with fluency and wit, and he is a good story-teller. He parses the mathematical universe into Algebra, Geometry, Applied Mathematics and Analysis, and gives each its share. The book is structured into four parts (corresponding to historical periods), each with six chapters. Each chapter has a key equation, ranging from $1 + 1 = 2$ to $2^{\aleph_0} = \aleph_1$, and including physical equations from

Received on 24-8-2012.

©2012 Irish Mathematical Society

Archimedes, Newton, Maxwell, Einstein and Dirac. This framework provides the skeleton for a tour through the whole history of mathematical ideas and characters, including Pythagoras, Cardano, Kepler, Euler, Abel, Galois, Gauss, Lobachevskii, Hamilton, Riemann and Chern. In the modern period, he includes a chapter on the Lorenz attractor and an account of Black-Scholes and its influence on financial markets. The book is beautifully-illustrated with many full-colour pictures.

There is a wealth of anecdote and information. It was amusing to learn that a 1988 poll taken for the Mathematical Intelligencer revealed that Euler proved four of the top five "most beautiful theorems", and that a similar exercise for Physics World showed Maxwell's Equations to be the most popular.

The author draws morals from the tales, and expresses clear opinions. This makes for interesting reading, although one is obliged to take issue on some points:

He gives a proof by diagram of Pythagoras taken from Liu Hui's third-century annotation of the classic *Nine Chapters on the Art of Mathematics*, and remarks (p. 39) that it is "a much simpler proof to understand than the one in Euclid's *Elements*". It is an interesting diagram, but it is not actually a proof, as it implicitly assumes the congruence of various pairs of figures.

I was interested to learn that this same Liu gave the approximation $\pi \approx \frac{3927}{1250}$, based on the use of an equilateral 3072-sided polygon, extending the method used (presumably independently) by Archimedes. On p. 45 he concludes an interesting account of formulae for π, including one that gives the octal expansion with the quite fatuous statement: "If God created the integers and God created π, then perhaps God is actually a computer".

On p. 47 he states that "To the modern mathematician, Zeno's paradoxes are harmless". In this, he certainly is in accord with many who underrate Zeno. But Zeno was talking about Physics, the world, and he gave *three* paradoxes that must be considered together. I would say that they are resolvable, but not that they are harmless.

On p. 51, discussing the area under a parabola, he outlines the method given by Archimedes, and compares it unfavourably with the procedure that would be followed by a "modern mathematician, who would have no qualms about" taking a limit in a certain series.

But in fact the careful argument used by Archimedes, beginning by assuming that the area is not what is asserted and proceeding by *reductio ad absurdum* corresponds *precisely* to what a modern (rigorous) mathematician would do: if the area is not $\frac{4}{3}$, then it is greater or less by some amount $\epsilon > 0$, and so on.

Chapter 6, on Archimedes law for levers, includes a statement of his famous Principle about immersed bodies. This Principle is often introduced in school physics courses, as it is here, without any discussion of the reasoning behind it. This is a pity, as it is easily proved in the case of a rectangular box, using elementary arithmetic and the nature of pressure. The case of a body of arbitrary shape requires multivariate integral calculus, or equivalent. It would be inappropriate to include it in the present book, and I do not fault it for that.

On p.73, Kepler's Third Law is mis-stated, using the "distance to the Sun" instead of the semi-axis major. The difference is often trivial enough, but there seems no good reason not to give the correct version. More seriously, there are some questionable statements about the consequences of this Law. It is not true to say that one can infer the distance of an orbiting planet from its period (unless one knows the mass of the star or the distance and period of some other planet). It is not true that one can tell the mass of a planet from the observation of its period — although one can tell it from the period and distance of one of its satellites.

On p. 149, it is implied that the dynamical pressure of the solar wind is the same thing as the pressure of solar radiation.

On p. 185 he gives a proof that the set of real numbers is uncountable, using decimal expansions, remarking in footnote that he "Intentionally gave this easier but flawed version for non-experts. For math experts, repairing the inaccuracy takes a little work but in my opinion no fundamentally new ideas." However, the proof may be fixed in a very simple way without making it any more difficult for non-experts: instead of adding $1 \mod 10$ to the n-th digit of the n-th decimal in a purported enumeration of the interval $[0, 1)$, just change it to a 2 if it is not a 2, and otherwise change it to a 3.

These quibbles are matters that can easily be fixed in a new edition, and once that is done I would be happy to recommend the book to any person, young or old, with an interest in mathematics and its uses. I enjoyed reading it.

Anthony G. O'Farrell was educated at UCD and Brown University. He is Professor of Mathematics at NUI, Maynooth. His research interests centre on Analysis.

Department of Mathematics and Statistics, NUI, Maynooth, Co. Kildare

E-mail address: admin@maths.nuim.ie

Irish Math. Soc. Bulletin
Number 67, Summer 2012, 67–70
ISSN 0791-5578

PROBLEMS

IAN SHORT

The first two problems were contributed by Finbarr Holland.

Problem 69.1. Suppose that the matrices A, b, and c are of sizes $n \times n$, $n \times 1$, and $1 \times n$, respectively. Prove that, for all complex numbers z,

$$\det(A - zbc) = \det A - zcA^*b = \det A + z\det\begin{pmatrix} 0 & c \\ b & A \end{pmatrix},$$

where A^* is the adjoint of A (that is, the transpose of the matrix of cofactors of A).

Problem 69.2. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \sum_{k=1}^{n} \frac{1}{k} = \zeta(3),$$

where ζ is the Riemann zeta function.

I came across the final problem as a graduate student.

Problem 69.3. A rectangle is partitioned into finitely many smaller rectangles. Each of these smaller rectangles has a side of integral length. Prove that the larger rectangle also has a side of integral length.

Here are the solutions to the problems from *Bulletin* Number 67. The first solution was contributed by the North Kildare Mathematics Problem Club.

Problem 67.1. Prove that there does not exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$ that satisfies

$$f'(x) \geqslant 1 + [f(x)]^2$$

for each real number x.

Received on 24-8-2012.

©2012 Irish Mathematical Society

Solution 67.1. Suppose there were such an f. Let

$$g(x) = \arctan f(x) - x.$$

Then

$$g'(x) = \frac{f'(x)}{1 + f(x)^2} - 1 \geqslant 0$$

for all $x \in \mathbb{R}$, so g is nondecreasing, so for $x > 0$ we have

$$\arctan f(x) \geqslant x + \arctan f(0).$$

This is impossible, because $\arctan f(x) < \pi/2$ for all real numbers x. Therefore no such function f exists. □

The second solution was shown to me some years ago by Edward Crane, shortly after he was given the problem.

*Problem 67.2***.** Suppose that $x_1, x_2, \ldots, x_n$, where $n \geqslant 3$, are non-negative real numbers such that

$$x_1 + x_2 + \cdots + x_n = 2$$

and

$$x_1x_2 + x_2x_3 + \cdots + x_{n-2}x_{n-1} + x_{n-1}x_n = 1.$$

Find the maximum and minimum values of

$$x_1^2 + x_2^2 + \cdots + x_n^2.$$

Solution 67.2. Let

$$A = \sum_{\substack{i \leqslant n \\ i \text{ odd}}} x_i \quad \text{and} \quad B = \sum_{\substack{i \leqslant n \\ i \text{ even}}} x_i.$$

Then $A + B = 2$, so $AB \leqslant 1$, and hence

$$1 = x_1x_2 + x_2x_3 + \cdots + x_{n-2}x_{n-1} + x_{n-1}x_n \leqslant AB \leqslant 1.$$

Equality in the first inequality implies that all terms x_j are 0 other than three consecutive terms x_{i-1}, x_i, and x_{i+1} (and one of these may be 0). This reduces the problem to the $n = 3$ case. In this case you can easily check that $x_2 = 1$, and the minimum is $3/2$ and the maximum is 2. □

The third solution was contributed by the North Kildare Mathematics Problem Club (they also submitted an alternative solution to the second problem).

*Problem 67.3***.** There are m gold coins divided unequally between n chests. An enormous queue of people are asked in turn to select a chest. Each member of the queue knows how many coins there are in each chest, and also knows the choice of those ahead in the queue who have selected already. In choosing a chest, each person considers the (possibly non-integer) number of gold coins he would receive were the coins in that chest to be shared equally amongst all those, including him, who have selected that chest so far. He then chooses the chest that maximises this number of coins. For example, if there are three chests A, B, and C containing 3, 5, and 8 coins, then the first person in the queue selects C, the second selects B, the third selects C, the fourth selects A, and so forth.

After the mth person has chosen a chest, how many people have selected each chest? Express your answer in terms of the number of coins per chest. What more can be said about people's chest selections?

Solution 67.3. We claim that the number of people who choose each chest by the m-th stage is equal to the number of coins in the chest.

We remark that the choice of chest is not always uniquely determined. However, this does not affect the state of play after m choices.

Suppose c_j coins are in chest j. Let $p_j^0 = 0$ for all j. For $n \geq 1$, let p_j^n be the number who have chosen chest j when the n-th person has made his choice.

Our claim is that $p_j^m = c_j$ for each j.

Suppose some $p_k^m > c_k$. Let n be the first number with $p_k^n > c_k$. Then

$$\sum_{j \neq k} p_j^n = n - p_k^n < m - c_k = \sum_{j \neq k} c_j,$$

so there exists some j with $p_j^n < c_j$. But then chest j would have been a better choice than chest k at the n-th stage, since

$$\frac{c_j}{p_j^n + 1} \geqslant 1 > \frac{c_k}{p_k^n}.$$

So this is impossible.

Thus $p_j^m \leqslant c_j$ for each j, and since

$$m = \sum_j p_j^m \leqslant \sum_j c_j = m,$$

we conclude that $p_j^m = c_j$ for each j, as claimed.

More generally, regarding each coin as a packet of r coins, we see that when $n = mr$, the number choosing chest j is rc_j. $\square$

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com`.

Department of Mathematics and Statistics, The Open University, Milton Keynes MK7 6AA, United Kingdom

Instructions to Authors

The *Bulletin* is typeset using LaTeX; therefore, authors should submit articles to the *Bulletin* as LaTeX input files together with any macro files or input files for the production of graphics. Typescripts are **not** acceptable. Source files should be suitable for processing using `pdflatex`. All files should be accompanied by any non-standard style or input files which have been used. Private `METAFONT` and TEX macros, reference input files and files for diagrams must also accompany submissions. The preferred method of submission is by e-mail to

`mailto://ims.bulletin@gmail.com`

Only if this is not possible, authors may send a CD with their files to the Editor together with a printout of the typescript. If articles are submitted in electronic form, no hard copies need to be sent unless requested by the Editor. Once a paper is accepted in final form, the author(s) will be responsible for producing an output according to the *Bulletin's* standard layout. To this end, authors will be supplied with style files. Since normally no proofs are sent out before publication, it is the author's responsibility to check carefully for any misprints or typos. After publication, authors will be sent a free copy of the issue in which their article(s) appeared as well as a pdf file containing their article(s). The Society grants authors free use of *this* pdf file; hence they may post it on personal web sites or electronic archives.