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EDITORIAL

Since this is the penultimate issue of the Bulletin *that I shall be editing, I use this opportunity to reflect on what could/should go into this publication of the IMS. The stated aim of the* Bulletin *is "to inform Society members, and the mathematical community in large, about the activities of the Society and about items of general mathematical interest." Until some time ago, announcements of mathematical meetings were included; but it was agreed that a printed publication nowadays is far too slow a vehicle for such a purpose and, moreover, very few conference organisers would send in their information well in time to be included in the* Bulletin*. Subsequently, organisers of meetings that obtained support from the IMS were asked to write up a short summary of the outcomes of their meetings and submit these to the* Bulletin*; understandably, this never got to work—who bothers about the details of a conference after it is held? There is moderate success in reporting on the annual (September) meeting of the IMS—which clearly should fall under the above-stated aim—but the editor really depends on the help of the organisers in order to get the abstracts of the talks etc. in (and in time for the winter issue!).*

The section on abstracts of PhD theses became more popular (I even had some interest from outside Ireland to submit an abstract!) so I hope this will continue to flourish. (Here, the responsibility rests with the supervisors to encourage their students to submit an abstract (in time).) Compared with this, the Book Reviews *have not taken off, despite the fact that there is one in this volume. Far too few people nowadays have an interest to invest the time of reading a mathematical book in such detail that they could write a critical review on it (and publish it in a very small periodical).*

It is a pity, in my opinion, that the Departmental News *never became really attractive enough in order to catch the Departments'*

Heads' attention to submit the relevant information. Fortunately, over the years, there were a number of initiatives taken by (potential) authors themselves to submit a contribution to the Bulletin *or enquire whether something might be suitable. An excellent example is contained in this volume's* Miscellenea*, an account on a mathematical event that is intimately tied to Irish mathematics and history and is very much alive today. The editor says thanks a million to everyone who came up with an idea like this one in the past!*

This leaves us with the surveys and research notes. It is indeed the editor's job to solicit good and suitable survey articles "of interest not only to a small section of the mathematical community". While I put quite a bit of effort into finding and persuading people to compose such pieces of mathematical work until recently, I came to realise that fewer and fewer mathematicians want to write substantial surveys on their and their fellows' work in a style accessible to the non-specialist. A general trend, I believe, and, once again, the fact that the Bulletin *is a small, not widely known journal does not help.*

I also tried to attract good short research papers and must admit that my success has been limited. It appears that a number of authors consider a small periodical as an outlet for mediocre papers they cannot publish elsewhere. This resulted in a volume of submissions that often are even not worth to be sent to a referee. Luckily there were reasonable numbers of good papers coming in so far, but it is a price to pay—by the editor: if there were no research notes, there would be far fewer to reject! Well, I leave these decisions to better hands in the future.

—MM

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Korn Inequalities in Orlicz Spaces

MARTIN FUCHS

Abstract. We use gradient estimates for solutions of elliptic equations to obtain Korn's inequality for fields with zero trace from Orlicz–Sobolev classes.

As outlined for example in the monographs of Málek, Nečas, Rokyta, Růžička [18], of Duvaut and Lions [7] and of Zeidler [26], the well-posedness of many variational problems arising in fluid mechanics or in the mechanics of solids heavily depends on the positive answer to the following question: given a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary $\partial\Omega$ and a field $u : \Omega \to \mathbb{R}^n$ with zero trace, is it possible to bound a suitable energy norm (being determined by the variational problem under consideration) of the Jacobian matrix $\nabla u = \left(\partial_\alpha u^i\right)_{1\leq\alpha,i\leq n}$ in terms of the norm of the symmetric gradient $\varepsilon(u) := \frac{1}{2}\left(\partial_\alpha u^i + \partial_i u^\alpha\right)_{1\leq\alpha,i\leq n}$? Estimates of this form are known as Korn type inequalities, and the most elementary variant reads as follows: for any function u from the Sobolev space $\overset{\circ}{W}{}^1_2(\Omega;\mathbb{R}^n)$ (see Adams [2] for a definition) it holds

$$\int_\Omega |\nabla u|^2\,dx \leq 2\int_\Omega |\varepsilon(u)|^2\,dx\,. \tag{1}$$

In fact, if u denotes a smooth function with compact support in Ω, then (1) can be obtained by partial integration, and the validity of (1) for Sobolev functions with zero trace is immediate. We note that L^2-variants of Korn's inequality go back to the works of Courant and Hilbert [5], Friedrichs [9], Èidus [8] and Mihlin [19].

Next we pass to the L^p-case with exponent $1 < p < \infty$. Then it was shown by Gobert [Go1,2], Nečas [21], Mosolov and Mjasnikov [20], Temam [24] and later by the author [10] that there exists a

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positive constant $K = K_{p,n}(\Omega)$ depending on p and the dimension n as well as on the domain Ω such that the inequality

$$\int_\Omega |\nabla u|^p \, dx \le K \int_\Omega |\varepsilon(u)|^p \, dx \tag{2}$$

is fulfilled for all $u \in \overset{\circ}{W}{}^1_p(\Omega;\mathbb{R}^n)$. A nice proof of (2) is presented in Theorem 1.10, p.196, of [18] based on results of Nečas about equivalent norms on $L^p(\Omega;\mathbb{R}^n)$ in terms of negative norms, which correspond to norms on the dual Sobolev space $\overset{\circ}{W}{}^1_p(\Omega;\mathbb{R}^n)^*$.

Our short note now has been inspired by Remark 5 in the paper of Mosolov and Mjasnikov [20], where it is stated that "using the theorems of Simonenko [23] and Koizumi [15, 16] regarding the continuity of singular operators in Orlicz spaces, one can prove inequalities of the type of Korn's inequality in the corresponding spaces". To be precise, let us introduce the class Φ of all functions $\varphi : [0,\infty) \to [0,\infty)$, which are increasing and satisfy $\lim\limits_{t\downarrow 0}\varphi(t) = 0$, $\lim\limits_{t\to\infty} \varphi(t) = \infty$.

Definition 1. A function $\varphi \in \Phi$ is a *Young function* if φ is convex together with $\lim\limits_{t\downarrow 0} \varphi(t)/t = \lim\limits_{t\to\infty} t/\varphi(t) = 0$.

Definition 2. Let φ denote a Young function.

a) φ is *of type* $(\Delta 2)$ if there is a constant $K > 0$ such that $\varphi(2t) \le K\varphi(t)$ holds for all $t \ge 0$.

b) We define φ to be *of type* $(\nabla 2)$ if for some constant $\widetilde{K} > 1$ we have $\varphi(t) \le \frac{1}{2\widetilde{K}}\varphi(\widetilde{K}t)$, $t \ge 0$.

Remark 1. The $(\Delta 2)$ property—also known as doubling property of the Young function φ—guarantees that the Orlicz class $K_\varphi(\Omega)$ and the Orlicz space $L_\varphi(\Omega)$ coincide (see [2], Chapter VIII, for notation). Moreover, we can introduce the Orlicz–Sobolev space $\overset{\circ}{W}{}^1_\varphi(\Omega;\mathbb{R}^n)$ of functions with zero trace in the usual way.

Remark 2. It should be noted that both $(\Delta 2)$ and $(\nabla 2)$ conditions make the Young function grow moderately. For example, if φ is of class C^1, then $(\nabla 2)$ follows from the requirement that

$$a(\varphi) := \inf_{t>0} \frac{t\varphi'(t)}{\varphi(t)} > 1$$

is fulfilled. We refer the reader to the monograph of Roa and Ren [22], Corollary 4 on p.26.

With this notation we can give the following interpretation of Remark 5 from the paper [20].

Theorem 1. *Let Ω be a bounded Lipschitz domain with small Lipschitz constants. Let φ denote a Young function of type $(\Delta 2)\cap(\nabla 2)$. Then there is a constant $C = C_{\varphi,n}(\Omega)$ depending on φ, the dimension n and the domain Ω such that*

$$\int_\Omega \varphi(|\nabla u|)\, dx \le C \int_\Omega \varphi(|\varepsilon(u)|)\, dx \tag{3}$$

is true for all fields $u \in \overset{\circ}{W}{}^1_\varphi(\Omega;\mathbb{R}^n)$. In the case of two independent variables (3) *can be replaced by*

$$\int_\Omega \varphi(|\nabla u|)\, dx \le C \int_\Omega \varphi(|\varepsilon^D(u)|)\, dx\ , \tag{4}$$

where $\varepsilon^D(u) := \varepsilon(u) - \frac{1}{n}(\operatorname{div} u)\mathbf{1}$ is the deviatoric part of $\varepsilon(u)$, $\mathbf{1}$ denoting the unit matrix.

Remark 3. In their deep paper on stationary electrorheological fluids Acerbi and Mingione [1] prove variants of (3) for some special Young functions φ (see Theorem 3.1 in this reference). Their argument is based on a kind of representation formula due to Ambrosio, Coscia and Dal Maso [3] (using in turn information from Kohn's thesis [14]) combined with an interpolation argument originating from Torchinsky's work [25].

Proof of Theorem 1. Our proof of (3) and (4) is based on gradient estimates in Orlicz spaces for solutions of elliptic equations recently obtained by Jia, Li and Wang [13]. Let u denote a function from the class $C_0^\infty(\Omega;\mathbb{R}^n)$, the general case follows by approximation. Then, as observed by Dain [6], we have the formula

$$\Delta u^j = 2\partial_i \varepsilon^D(u)_{ij} - 2\left(\frac{1}{2} - \frac{1}{n}\right)\partial_j(\operatorname{div} u), \quad j = 1, \dots, n\,, \tag{5}$$

where the sum is taken with respect to indices repeated twice. We fix a coordinate direction $j \in \{1, \dots, n\}$ and define $v := u^j$,

$$V := \left(2\varepsilon^D(u)_{ij} - 2\left(\frac{1}{2} - \frac{1}{n}\right)\operatorname{div} u\, \delta_{ij}\right)_{1\le i\le n}.$$

Then, according to (5), $\Delta v = \operatorname{div} V$ and the desired inequalities (3) and (4) are a direct consequence of estimate (3.1) in Theorem 3.1 of [13], since on account of this reference we have the bound $\int_\Omega \varphi(|\nabla v|)\,dx \le C\int_\Omega \varphi(|V|)\,dx$. Note that the domain Ω satisfies the assumptions from [13], as clarified in Remark 5 below. □

Remark 4. We remark that Krylov [17] obtained Korn's inequality in the L^p-setting ($1 < p < \infty$) for fields equal to zero on the boundary by similar arguments as a result of his studies of the regularity properties of solutions to the Dirichlet problem for the Poisson equation in norms of negative order.

Remark 5. Of course the estimates (3) and (4) extend to the class of Reifenberg domains $\Omega \subset \mathbb{R}^n$ studied in the paper [13]. This follows from the comments given by Byun, Yao and Zhou stated after inequality (1.4) in their work [4], where it is said that domains with sufficiently small Lipschitz constants are (δ, R)-Reifenberg flat.

Remark 6. As outlined in [13] and also discussed by Byun, Yao and Zhou [4] the basic gradient estimate for the Poisson equation with Orlicz space data holds if and only if the Young function φ is of type $(\Delta 2) \cap (\nabla 2)$. This gives rise to the interesting question if $\varphi \in (\Delta 2) \cap (\nabla 2)$ is also a necessary condition for the validity of (3) and (4).

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Some Residually Solvable One-Relator Groups

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Abstract. This communication records some observations made in the course of studying one-relator groups from the point of view of residual solvability. As a contribution to classification efforts we single out some relator types that render the corresponding one-relator groups residually solvable.

1. Introduction

It is well known that free groups are residually nilpotent and, consequently, residually solvable. The literature contains a sizable amount of information about structural, residual, virtual properties of one-relator groups. The purpose of this communication is to offer a collection of facts and examples gathered while attempting to characterize the residually solvable one-relator groups in terms (of the form) of the (single) defining relator. In what follows we prove sufficiency results for certain cases when the relator is a commutator, and then raise some questions.

The class of one-relator groups shows a varied pattern of behavior with respect to residual properties. We begin with reviewing some of the literature that motivated our interest in the topic. G. Baumslag in [3] showed that positive one-relator groups, which is to say that the relator has only positive exponents, are residually solvable. In the same paper he provided a specific example to demonstrate that not all one-relator groups are residually solvable. A free-by-cyclic group is necessarily residually solvable. As well are the free-by-solvable Baumslag–Solitar groups $B_{m,n}$ (the groups with presentation $\langle a, b; a^{-1}b^m a = b^n\rangle$ for pairs of non-zero integers m, n), by a result of Peter Kropholler [15] who showed that in these groups the second derived subgroup is free. The Baumslag–Solitar groups

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$B_{1,n}$ ($n \neq 0$ integer) are solvable but not polycyclic; and the non-Hopfian (therefore not residually finite) Baumslag–Solitar groups are not residually nilpotent.

It is worth mentioning here that a large class of residually solvable one-relator groups is indicated by [8]: G. Baumslag, Fine, Miller and Troeger established that many one-relator groups, in particular cyclically pinched one-relator groups, are either free-by-cyclic or virtually free-by-cyclic. Further, a recent result of M. Sapir and I. Spakulova in [17] tells that, with probability (measured in terms of the length of the relator) tending to 1, a one-relator group with at least 3 generators is residually finite, even virtually a residually (finite p)-group, and coherent, for all sufficiently large primes p. In his subsequent work [16] M. Sapir also focuses on residual properties of one-relator groups with at least 3 generators. Our attention is turned mainly toward two-generator one-relator groups.

Our two main results concern the situation where G is a one-relator group whose relator is a commutator. First we recall, in fair detail, some classes of one-relator groups whose behavior with regard to residual solvability has been established before. Then we show sufficiency, for residual solvability, of certain conditions imposed on the (single) defining relator of the one-relator group. Then, we provide examples illustrating the difficulty in determining residual solvability of one-relator groups with arbitrary commutator.

Clearly, attempts to find criteria for residual solvability could be facilitated by linkages to outcomes of recent and older studies on the (fully) residually freeness of one-relator groups, in particular. For example, surface groups are easily found residually solvable since they are known to be fully residually free [6]. Also, in [2] B. Baumslag shows residual freeness for one-relator groups of the type $\langle a_1, \cdots, a_k; {a_1}^{w_1} \cdots {a_k}^{w_k} = 1\rangle$, where $k > 3$, and the w_i's in the ambient free group on $a_1, \cdots, a_k$ satisfy certain conditions; thus residual solvability for such one-relator groups is immediate.

2. Preliminaries

For convenience, we start with a list of some of the definitions, facts, and theorems we will rely on throughout.

Theorem 2.1 (Von Dyck)**.** *Suppose $G = \langle X; R\rangle$ and $D = \langle X; R \cup S\rangle$, with presentation maps γ and μ respectively. Then $x\mu \mapsto x\gamma$ $(x \in X)$ defines a homomorphism of G onto D.*

Theorem 2.2 (Freiheitssatz, [14])**.** *Let G be a one-relator group, i.e., $G = \langle x_1, \cdots, x_q; r = 1\rangle$. Suppose that the relator r is cyclically reduced, i.e., the first and the last letters in r are not (formal) inverses of each other. If each of $x_1, \cdots, x_q$ actually appears in r, then any proper subset of $\{x_1, \cdots, x_q\}$ is a free basis for a free subgroup of G.*

W. Magnus' method of structure analysis [14] for groups with a single defining relation has the following immediate consequence:

Lemma 2.3. *Let $G = \langle b, x, \cdots, c; r = 1\rangle$ be a one-relator group. Suppose that b occurs in r with exponent sum zero and that upon re-expressing r in terms of the conjugates $b^i x b^{-i} = x_i, \cdots, b^i c b^{-i} = c_i$ $(i \in \mathbb{Z})$ and renaming r as r_0, μ and ν are respectively the minimum and maximum subscripts of x occurring in r_0. If $\mu < \nu$ and if both x_μ and x_ν occur only once in r_0 then $N = gp_G(x, \cdots, c)$ is free. If G is a two-generator group with generators b and x, then N is free of rank $\nu - \mu + 1$.*

Definition 2.4. A group G is *residually solvable* if for each $w \in G$ $(w \neq 1)$, there exists a solvable group $S = S(w)$ and an epimorphism $\phi : G \longrightarrow S$ such that $w\phi \neq 1$.

Theorem 2.5 (Kahrobaei, [12], [13])**.** *Any generalized free product of two finitely generated torsion-free nilpotent groups, amalgamating a cyclic subgroup, is an extension of a residually solvable group by a solvable group. It is therefore residually solvable.*

Theorem 2.6 (Kahrobaei, [12], [13])**.** *Any generalized free product of an arbitrary number of finitely generated nilpotent groups of bounded class, amalgamating a subgroup central in each of the factors, is an extension of a free group by a nilpotent group. It is therefore residually solvable.*

Theorem 2.7 (Kahrobaei, [12], [13])**.** *The generalized free product of a finitely generated torsion-free abelian group and a nilpotent group is (residually solvable)-by-abelian-by-(finite abelian), consequently residually solvable.*

Note that the groups in all three of these theorems above satisfy the conditions of K. Gruenberg's portent observation [10] that we record here as

Lemma 2.8. *Suppose P is any group, $K \lhd P$ with P/K solvable and K residually solvable. Then P is residually solvable.*

3. The Single Relator is a Commutator

We first recall a result from [4] for a particular class of non-positive one-relator groups. Let G be a group that can be presented in the form,

$$G = \langle t, a, ..., c; uw^{-1} = 1\rangle, \tag{1}$$

where u and w are positive words in the given generators and each generator occurs with exponent sum zero in uw^{-1}. Then G is residually solvable. In fact, G is free-by-cyclic.

Now consider the group

$$H = \langle t, a, ..., c; [u, w] = 1\rangle. \tag{2}$$

If u and w are positive, H can be recognized as one of the groups in the preceding class (1). Hence H is free-by-cyclic and therefore residually solvable. However, known examples show that residual solvability for H is not guaranteed once the requirement that both u or w be positive is relaxed:

Example 3.1. [4] If $G = \langle a, b, ..., c; [u, v] = 1\rangle$, where

$$u = a,\ v = [a, b][w, w^b],\ \text{and}\ w = [a, b]^{-1}[a, b]^a,$$

then G is not residually solvable.

Proof. It follows from Magnus' solution of the word problem that $w \neq 1$ in G [14]. Since $[u, v] = 1$, we find that

$$[a, b]^a[w, w^b]^a = [a, b][w, w^b],$$

so that

$$w = [a, b]^{-1}[a, b]^a = [w, w^b]([w, w^b]^a)^{-1}.$$

Thus w lies in every term of the derived series of G. □

In contrast, the next example is a residually solvable one-relator group.

Example 3.2. The group $G = \langle a, b; [a, [a, b]]\rangle$ is free-by-cyclic.

Proof. We expand and re-express the relator,

$$r = [a, [a, b]] = a^{-1}[a, b]^{-1}a[a, b] = a^{-1}b^{-1}a^{-1}bab^{-1}ab. \tag{3}$$

Observe that in

$$r_0 = b_1^{-1}b_2b_1^{-1}b_0$$

$\mu = 2$, $\nu = 0$, b_0 and b_2 both occur only once, so we can invoke Lemma 2.3. Therefore G, as a cyclic extension of the free group $N = gp_G(b)$, is residually solvable (cf. 2.8). □

4. Connection between Generalized Free Products and One-Relator Groups

Over the years since W. Magnus developed his treatment of one-relator groups the increased interest in them yielded many new results. Karrass and Solitar in 1971 showed that a subgroup of a one-relator group is either solvable or contains a free subgroup of rank two. G. Baumslag and Shalen showed that every one-relator group with at least four generators can be decomposed into a generalized free product of two groups where the amalgamated subgroup is proper in one factor and of infinite index in the other. Fine, Howie and Rosenberger [7], and Culler and Morgan [9] showed that any one-relator group with torsion and at least three generators can be decomposed, in a non-trivial way, as an amalgamated free product. These results made it seem reasonable to expect that a closer look at the residual solvability of generalized free products of two groups could provide further tools for detection of residual solvability of one-relator groups. The following result confirms that assumption.

Theorem 4.1. *The group* $G = \langle a, b; [a, w]\rangle$, *where* $w = [a, b]^n$ $(n \in \mathbb{N})$, *is residually solvable.*

Proof. Put $N = gp_G(b)$, the normal closure of b in G. Using the Magnus break-down, we consider:

$$N_0 = \langle b_0, b_1, b_2; (b_1b_0)^n = (b_2^{-1}b_1)^n\rangle. \tag{4}$$

Now let

$$x_0 = b_1^{-1}b_0,\ x_1 = b_2^{-1}b_1,\ y = b_1.$$

Tietze transformations confirm that

$$N_0 = \langle x_0, x_1, y; (x_0)^n = (x_1)^n\rangle = \langle x_0, x_1; (x_0)^n = (x_1)^n\rangle * \langle y\rangle. \tag{5}$$

Next let $K = \langle x_0, x_1; (x_0)^n = (x_1)^n\rangle$. Clearly

$$K = \{\langle x_0\rangle * \langle x_1\rangle; \langle x_0^n\rangle = \langle x_1^n\rangle\} \tag{6}$$

Since each factor of K is abelian, by Theorem 2.5 K is residually solvable. The free factor of N_0, $\langle y\rangle$ is also residually solvable. Therefore N_0 is residually solvable, and it follows for every $i \in \mathbb{N}$ that N_i is residually solvable. If we put $N_{i,j} = gp(N_i, N_{i+1}, ..., N_j)$, the preceding approach gives

$$N_{i,j} = \langle x_i\rangle *_{\langle x_i^n\rangle = \langle x_{i+1}^n\rangle} *\langle x_{i+1}\rangle * \cdots * \langle x_j\rangle *_{\langle x_j^n\rangle = \langle x_{j+1}^n\rangle} *\langle x_{j+1}\rangle * \langle y\rangle.$$

Therefore, by Theorem 2.6, $N_{i,j}$ is residually solvable. A task that remains for completing the proof is to show that the ascending union $N = \cup_{r<0;s>0} N_{r,s}$ is residually solvable, which will be taken care of by the following Proposition 4.2. Granted that, the residual solvability of G follows with the use of Corollary 2.8. □

Proposition 4.2. $N = \cup_{r<0;s>0} N_{r,s}$ *is residually solvable.*

Proof. We will retain notation from the proof of Theorem 4.1 and start with the assumption that $N_{i,j}$ is residually solvable for all $i, j \in \mathbb{N}$ $(i \leq j)$. For the derived series of $N = \cup_{r<0;s>0} N_{r,s}$, we have

$$\delta_i N = \delta_i(\cup_{r<0;s>0} N_{r,s}).$$

Every element $g \in \delta_i N$ is a finite product of commutators of elements from a (finite) subset of the $N_{r,s}$ groups. So $g \in \delta_i N_{r,s}$ for suitably small value of $r < 0$ and suitably large value of $s > 0$. Thus $\delta_i N = \cup_{r<0;s>0} N_{r,s}$. Now, if j, k are a pair of fixed integers the infinite union above can be rewritten as

$$\delta_i N = \cup_{r<0;s>0} \delta_i N_{j+r,k+s},$$

so that,

$$\delta_i N \cap N_{j,k} = (\cup_{r<0;s>0} \delta_i N_{j+r,k+s}) \cap N_{j,k}.$$

Equivalently,

$$\delta_i N \cap N_{j,k} = \cup_{r<0;s>0} (\delta_i N_{j+r,k+s} \cap N_{j,k}).$$

Further, each term in the union can be written as

$$\delta_i N_{j+r,k+s} \cap N_{j,k} = (\delta_i N_{j,k+s} \cap N_{j,k}) \cap (\delta_i N_{j+r,k} \cap N_{j,k}).$$

And because $s < 0$ and $r > 0$, an argument fashioned after that in [3, p. 175, Lemma 4.3] yields after suitable conjugations that

$$\delta_i N_{j,k+s} \cap N_{j,k} = \delta_i N_{j,k} \text{ and}$$

$$\delta_i N_{j+r,k} \cap N_{j,k} = \delta_i N_{j,k}.$$

So each term in the union can be re-expressed as

$$\delta_i N_{j+r,k+s} \cap N_{j,k} = \delta_i N_{j,k}.$$

Notice that this expression is independent of r and s. Thus, we get

$$\delta_i N \cap N_{j,k} = \delta_i N_{j,k} \text{ ([3, pg. 175, line 16]).}$$

We claim that

$$\delta_i N \cap N_{j,k} = \delta_i N_{j,k}$$

implies that N is residually solvable. To see this, let g be a non-trivial element of

$$N = \cup_{r<0;s>0} N_{r,s}.$$

Then there is an integer $j = j(g) \in \mathbb{N}$ such that $g \in N_{-j,j}$. By our (inductive) hypothesis at the outset $N_{-j,j}$ is residually solvable. Consequently, there exists an integer $i \in \mathbb{N}$ such that $g \notin \delta_i N_{-j,j}$. Then, since $\delta_i N \cup N_{-j,j} = \delta_i N_{-j,j}$, we see that $g \notin \delta_i N \cap N_{-j,j}$. But $g \in N_{-j,j}$. So it must be the case that $g \notin \delta_i N$. Thus we have found a normal subgroup $\delta_i N \lhd N$ with the property that $g \notin \delta_i N$ and $N/\delta_i N$ is solvable. Hence N is residually solvable. □

5. The Relator is a Basic Commutator

The tools of the Magnus theory were of good use for proving residual solvability through gaining information about the structure of the two-generator one-relator groups where the relator is a particular type of basic commutator.

We begin with recalling P. Hall's [11] definition of the basic commutators (in terms of the free group F on $\{x_1, ..., x_q\}$) and their linear ordering (in terms of their *weights*).

Definition 5.1. Basic Commutators.

(1) The basic commutators of weight one with their linear order are $x_1 < x_2 < \cdots < x_q$; for their weights we write $wt(x_i) = 1$.
(2) Having defined the basic commutators of weight less than n, the basic commutators with weight n are of the form $c_n = [c_i, c_j]$ where c_i and c_j are all the basic commutators satisfying $wt(c_i) + wt(c_j) = n$, $c_i > c_j$, and such that if $c_i = [c_s, c_t]$, then $c_j \geq c_t$.

In the following we will use the notation $s_1 = x$ and $s_{k+1} = [s_k, y]$ for positive integers k.

Theorem 5.2. *The group $G = \langle x, y; r = [s_k, y]\rangle$ is free-by-cyclic, therefore residually solvable.*

Proof. Following the Magnus theory we put $x_i = y^{-i}xy^i$ for this two generator case. Using induction on the weight of the commutator and the relationship

$$[s_k, y] = s_k^{-1}(s_k)^y \ (k > 0),$$

we see that the minimum index and maximum index in r_0 are 0 and k, respectively, and both of x_0 and x_k occur only once in r_0. By Lemma 2.3 it follows, similarly to previous cases, that G is free-by-cyclic. $\square$

6. Open Problems

(1) Is it algorithmically decidable whether a one-relator group is residually solvable?

(2) Are one-relator groups generically residually solvable? In other word, are they in most cases residually solvable? I. Kapovich conjectured [5] that many one-relator groups are finitely generated free-by-cyclic.

(3) Do there exist residually finite one-relator groups that are not residually solvable? (This is recasting a question in [1] in this context.)

(4) As defining relators, certain basic commutators were shown in this paper to render the respective one-relator groups residually solvable. Would all basic commutators have that property? If not, can the techniques used here be extended to one-relator groups with further types of basic commutator for their defining relator?

(5) Find further examples of non-positive one relator groups that fail to be residually solvable.

(6) Find examples of residually solvable one-relator groups that are not free-by-cyclic.

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Twenty Years of the Hamilton Walk

Fiacre Ó Cairbre

1. Introduction

It all started on a bright Monday morning on October 16, 1843. The famous event was later described in a letter from Hamilton to his son, as follows:

Although your mother talked with me now and then, yet an undercurrent of thought was going on in my mind, which gave at last a result, whereof it is not too much to say that I felt at once an importance. An electric current seemed to close; and a spark flashed forth, the herald (as I foresaw, immediately) of many long years to come of definitely directed thought and work .... Nor could I resist the impulse – unphilosophical as it may have been – to cut with a knife on a stone of Brougham Bridge as we passed it, the fundamental formula...

The above piece describes Hamilton's famous creation of a strange new system of four–dimensional numbers called Quaternions, which are his most celebrated contribution to mathematics. "Number couples" (or complex numbers) had been important in mathematics and science for working in two–dimensional geometry and Hamilton was trying to extend his theory of number couples to a theory of "Number triples" (or triplets). He hoped these triplets would provide a natural mathematical structure and a new way for describing our three–dimensional world, in the same way that the number couples played a significant role in two–dimensional geometry. He was having a difficult time defining the multiplication operation in his quest for a suitable theory of triplets. The story goes that at breakfast time, his son would ask "Well, Papa, can you multiply triplets?" and Hamilton would reply "No, I can only add and subtract them". We now know why Hamilton was having such a difficult time because it's actually impossible to construct the suitable theory of triplets he was pursuing. Then, on October 16, 1843, Hamilton's mind gave birth to quaternions in a flash of inspiration, as he walked along the banks of the Royal Canal at Broombridge. In a nineteenth century act of graffiti, Hamilton scratched his quaternion formulas on the bridge as described in his own words above.

Hamilton realised that if he worked with "Number quadruples" and an unusual multiplication operation, then he would get everything he desired. He named his new system of numbers Quaternions because each number quadruple had four components. He had created a totally new structure in mathematics. The mathematical world was shocked at his audacity in creating a system of "numbers" that did not satisfy the usual commutative rule for multiplication ($ab = ba$). Hamilton has been called the "Liberator of Algebra" because his quaternions smashed the previously accepted convention that a useful algebraic number system should satisfy the rules of ordinary numbers in arithmetic. His quaternions opened up a whole new mathematical landscape in which mathematicians were now free to conceive new algebraic number systems that were not shackled by the rules of ordinary numbers in arithmetic. Hence, we may say that Modern Algebra was born on October 16, 1843 on the banks of the Royal Canal in Dublin. The event is now commemorated by a plaque at Broombridge which was unveiled by the Taoiseach, Eamon de Valera, in 1958.

In 1990, Anthony G. O'Farrell initiated an annual walk to commemorate Hamilton's creation of the quaternions. The annual Hamilton walk takes place on October 16 and participants retrace Hamilton's steps by starting at Dunsink Observatory, where Hamilton lived, and then strolling down to meet the Royal Canal at Ashtown train station. The walk then continues along the canal to the commemorative plaque at Broombridge in Cabra. The walk takes about forty–five minutes.

I organise the annual Hamilton walk which will celebrate its twentieth anniversary this year in 2010. So, I suppose you could say "Fiche bliain ag siúl"! The walk now attracts about 200 people from diverse backgrounds including staff and students from third level, second level and many from the general public. The walk is ideal for a mathematics outing for transition year students and teachers have said that the walk and the Hamilton story have had a very positive impact on students' perception of mathematics. The large number of participants from the general public also indicates that there is a substantial public interest in Hamilton and the walk. Furthermore, I receive many calls from the media (television, radio and newspaper) and other bodies every year expressing an interest in doing a piece on Hamilton and the walk. Consequently, Hamilton's story and the walk have appeared three times on television and many times on a variety of radio programmes and in lots of newspaper articles, and I have given many talks on Hamilton. I feel there are a variety of reasons why there is such a large public/media interest in Hamilton's story and the walk. In [4] I discuss these reasons and they appear under the heading of the "Big picture of mathematics". Some (not all) of the items from the big picture of mathematics are: stories, famous characters, history of mathematics, beauty, practical power and applications, motivation, Irish connections, drama, humour and outdoor activities. I have lots of experience promoting mathematics in the general public and in second/third level. From the positive feedback I get from these groups, I find that the "big picture" approach enhances the understanding, awareness and appreciation of mathematics among them. Hamilton and the walk is a great example of something that has all the big picture items mentioned in [4] and can change the perception of mathematics (for the better) among second level students and the general public.

The general public plays a significant part in mathematics education in second and third level because parents, decision makers and the media are all members of the general public and can exert great influence on the attitude of young people and society at large towards mathematics.

Many famous people have come on the walk. Andrew Wiles launched the walk in 2003 and the walk appeared on the six o'clock news on RTÉ 1 television that evening. Fields Medallists, Timothy Gowers, Efim Zelmanov and Nobel Prize winners in Physics, Murray Gell–Mann, Steven Weinberg and Frank Wilczek have also participated in the walk in recent times. Also, in 2005 Hamilton's great-great grandson, Mike O'Regan, came on the walk.

Cabra Community Council have made the walk into a very festive affair with a large banner about Hamilton draped across the bridge and stalls along the canal. The following quote from Aodhan Perry of Cabra Community Council in 2009 captures the positive impact of the walk on the Cabra community:

The walk has had a huge impact on the local community. In fact it has gone way beyond just being a walk because all the local school children and the community are extremely proud of Hamilton and their local connection with him. The walk really has touched the local people in a big way. The fact that famous mathematicians and Nobel Prize winners mingle with school children and the local community on the walk and at the bridge is a great experience. Also, not one but two local artists have been commissioned in recent times to do portraits of Hamilton which are then publicly displayed at the bridge during the walk.

Here is another quote from local Cabra resident, Jack Gannon:

On account of the walk, Hamilton is in the folk consciousness of the local people.

Anybody who is interested in coming on the walk should contact me.

2. Selection of Walks from Various Years

1990: On October 8, 1990, Anthony G. O'Farrell wrote a piece entitled "The Quaternion Walk". In that article he wrote:

The relative durability of the Royal Canal and its stonework makes the scene of Hamilton's discovery unique,and has led to a steady trickle of mathematicians to make the pilgrimage to Broombridge. It is now proposed to make an annual commemoration on the anniversary of the discovery. The day was the 16th of October, 1843 and so this year it falls on a Tuesday.

This is how the first Quaternion walk (now called the Hamilton walk) was initiated by Anthony G. O'Farrell. The walk took place on October 16, 1990. People gathered at Dunsink Observatory where Anthony G. O'Farrell gave a short talk on Hamilton and the famous event. Then, the group retraced Hamilton's steps to the plaque at Broombridge.

1993: This was the sesquicentenary of the creation of quaternions. A surprise awaited us when we arrived at Broombridge. Somebody was already there gazing at the plaque. He had travelled all the way from New York in order to celebrate the sesquicentenary at the bridge! Anthony G. O'Farrell then gave a speech at the bridge about Hamilton's creation of quaternions.

1999: I organised something extra this year. After the walk from Dunsink to Broombridge a bus brought us from the bridge to Trim, Co. Meath where we visited the house where Hamilton spent his youth and received his early education. This house, now called St. Mary's Abbey, is beautifully situated on the banks of the River Boyne across from the spectacular ruins of Trim Castle. From there we proceeded to a local establishment for dinner and an evening of entertainment, including musical pieces by some Maynooth students, a table quiz and a talk on Hamilton by me (well, I had walked the walk and so now I talked the talk!). Also, Ciarán Ó Floinn had baked a cake with the quaternion formulas on it and this went down very well as part of the dessert! Finally, after a very enjoyable evening was had by all, the bus brought us back to Maynooth. In all there were about forty–five people, mostly students and staff from NUI, Maynooth.

2002: The Nobel Prize winner for Physics, Murray Gell–Mann, launched the walk at Dunsink Observatory by giving a short talk on Hamilton's work and its applications in Physics. About 100 people participated in the walk.

2003: A large group of about 150 people participated in the walk. Andrew Wiles launched the walk by giving a short talk on Hamilton and mathematics at Dunsink Observatory. Also, the Minister of State, Brian Lenihan, addressed the group at the Observatory. Later Anthony G. O'Farrell gave a speech at Broombridge. There was a diverse group of people including staff and students from NUI, Maynooth, Trinity College, St. Patrick's College, Drumcondra, DIT Kevin St, UCD, Maynooth Post–Primary and St. Andrews, Booterstown. There were also members of Cabra Community Council and the Royal Canal Amenity Group. A variety of individuals, who had read about the walk in the media, also joined us on the day. Jack Gannon, who lives close to Broombridge, came on the walk, and inspired by the walk, he wrote a song called "The Ballad of Rowan Hamilton" later in the year. The words of Jack's ballad appear at the end of this article. A television crew covered the walk and it appeared on the RTÉ 1 six o'clock news that evening.

Fiacre Ó Cairbre, Andrew Wiles and
Anthony G. O'Farrell on the 2003 walk

2004: Fields Medallist, Timothy Gowers, gave a short talk in Dunsink Observatory at the start of the walk. Jack Gannon's song, The Ballad of Rowan Hamilton, was first performed at the bridge after Anthony G. O'Farrell's speech. Jack's song has been played many times on programmes about Hamilton and the walk on radio and television since.

2005: This was a special year for the walk as it was the bicentenary of Hamilton's birth. The Government designated the year as Hamilton Year – Celebrating Irish Science. Many events were held all over Ireland to celebrate Hamilton year. There was also a commemorative Hamilton stamp and coin. Léargas produced a 30 minute television documentary on Hamilton on RTÉ 1 and there was also an RTÉ radio 1 programme on Hamilton as part of the Icons of Science series. On August 4 the Cabra Community Council celebrated his birthday with a huge party and a barge trip along the canal.

A large group of about 150 people came on the walk. Nobel Prize winner, Steven Weinberg, launched the walk with a talk in Dunsink Observatory. Hamilton's great–great grandson, Mike O'Regan, participated in the walk. Hamilton's daughter, Helen, had married John O'Regan in 1869 but tragically died a year later following the birth of her son John. Mike is descended from John and now lives in England.

Anthony G. O'Farrell speaking at Broombridge on the 2005 walk

After Anthony G. O'Farrell's speech at Broombridge, the crowd joined in for a rendition of Jack Gannon's ballad about Hamilton. Then, June Robinson got up on the wall beside the bridge and gave the first recital of her poem, The Benefactor, which she had written about Hamilton earlier in the year. June's poem appears in [7]. I am aware of two other poems written about Hamilton. There was a wide variety of participants including staff and students from NUI, Maynooth, St. Patrick's College, Drumcondra, DCU and St. Columba's College. There were also members of Cabra Community Council and the Royal Canal Amenity Group. A large group of individuals, who had read about the walk in the media, also came on the walk. A television crew covered the walk and it appeared as part of the above mentioned 30 minute Léargas RTÉ 1 documentary on Hamilton on November 14.
Mick Kelly, from Swords, wrote the following:

The Hamilton walk was my licence to explore and express myself around the subject of mathematics. By the age of nine I had decided I couldn't do mathematics, but I had also developed a strong interest in things technical and scientific and this created a conflict that simmered in the background of my educational and professional career for forty years. That is until I took part in the Hamilton walk in 2005. That walk had a profound effect on me. Hearing not only a Nobel laureate and a professor of mathematics sing Hamilton's praises, but also local poets, school children, balladeers and the Cabra community council, spurred me to turn my desire to celebrate Ireland's Science

Heritage into action. That action turned out to be a family run business called Science Heritage Ireland selling placemats and coasters celebrating Hamilton.

Anthony G. O'Farrell, Steven Weinberg and Fiacre Ó Cairbre on the 2005 walk

2006: A very large group of about 250 people participated in the walk. Ingrid Daubechies gave a talk in Dunsink Observatory at the start of the walk. Later Maurice O'Reilly gave a speech at the bridge. There were so many people at the bridge that it required the presence of a few Gardaí to exert some crowd/traffic control because of the dangerous road nearby. This may have been the first time for a Police presence for crowd/traffic control at an outdoor mathematical event! I wandered over to one of the Gardaí to thank him for helping out. He then started to talk about Hamilton and remarked "Yeah, isn't Hamilton's maths used for space navigation nowadays". To which I replied "You're right". This incident is a good example of how Hamilton's story has spread throughout society at large. It reminds me of a similar recent event when a foreign mathematician took a taxi from Dublin airport to Maynooth for a conference. When the driver realised that his passenger was a mathematician, the driver talked about Hamilton pretty much the whole way to Maynooth. After arriving at the conference the surprised mathematician asked one of my colleagues "Are all Irish taxi drivers so knowledgeable about mathematics?"

There was a great variety of people on the walk including staff and students from NUI, Maynooth, St. Patrick's College, Drumcondra, Waterford IT, Maynooth Post–Primary, Lucan Community College, Maryfield College, Coláiste Bríde, St. Columba's College and St. Patrick's Cathedral Grammar School. Many other individuals, who had read about the walk in the media, also joined us on the day. Mary Mulvihill's radio crew covered the walk and it appeared on her RTÉ radio 1 programme later in the week.
The walk was one of the main events on the first day of the inaugural Maths Week Ireland which was initiated in 2006 by Eoin Gill of Calmast in Waterford IT. Maths Week occurs annually around the middle of October so that it includes October 16 and the walk. The aim of Maths Week is to promote mathematics among school children and the general public. I organise some events for Maths Week and the feedback from people has been very positive. Typically around fifty events take place during the week all over the country. Also, it's very heartening to see that, separate from the fifty or so events above, many schools now organise their own Maths Week events.

2007: Nobel Prize winner, Frank Wilczek, gave a short talk about quaternions in Dunsink Observatory at the start of the walk. His talk was quite entertaining and at one dramatic moment he took off his belt. For a few seconds the audience didn't know what was going to happen next! Then, he went on to use his belt to illustrate the non–commutativity of Hamilton's quaternions by twisting the belt at one end and then the other end etc. The walk was one of the main events for Maths Week Ireland. About 200 people came on the walk.
Mick Kelly, who had written about the 2005 walk earlier, wrote:

By the 2007 walk I could sense flaws developing in the glass wall I had built around learning mathematics and found it strange but very uplifting to be answering queries from people about quaternion algebra. There was a special sense of magic at Broombridge on that fine Tuesday October 16, 2007, when the canal bank was alive with children playing all kinds of mathematics games. I couldn't help but wonder how many bridges to the future the organisers of this walk and maths week had created for our children, one year into the Government's Strategy for Science, Technology and Innovation 2006–2013.

2008: Lisa Randall launched the walk in Dunsink Observatory with a short talk about Hamilton and Physics. A large group participated in the walk.

Lisa Randall speaking at Dunsink Observatory on the 2008 walk

2009: About 200 people participated in the walk. Fields Medallist Efim Zelmanov launched the walk in Dunsink Observatory with a short talk about Hamilton and the connection with his own research area in non-commutative algebras. I then talked about the famous event on October 16, 1843, Hamilton's pursuit of beauty in mathematics and some of the modern day applications of quaternions like computer games, special effects in movies etc. I also mentioned various items related to Hamilton that had arisen during the previous year. Three examples were:
(a) The Irish Times journalist, Dick Ahlstrom, had just finished, on that very day, a first draft of a fictional novel about Hamilton and he hopes to have it on the shelves by October 16, 2010.
(b) Maureen and Mick Kelly and I had submitted a proposal to the new Wax Museum in Dublin for a piece on Hamilton and consequently there is now a Hamilton display there.
(c) Richard Wilson's Elmgreen golf club, which is right beside the Observatory, now calls its Matchplay trophy the William Rowan Hamilton cup and Richard has also submitted a proposal for a Hamilton plaque at the nearest hole to the Observatory. This hole is (wait for it) the 16th hole! Furthermore, I think it's quite appropriate that

golfers should have to shout "fore" (think of quaternions) when they play the 16th hole with the Hamilton plaque!

The three examples above are further evidence of the spreading of the Hamilton story throughout society at large. Typically, I encounter a variety of new examples like these every year. Again, I feel this illustrates the large public/media appetite for Hamilton's story. On the day itself, the walk was featured on the Morning Ireland programme on RTÉ radio 1. Also, earlier in October the walk had been featured on the Capital D programme on RTÉ 1 television.

I also gave a special mention to the secondary school teacher, Roy Hession, and his pupils from St. Colman's in Claremorris, Co. Mayo. They now hold the record for the furthest distance travelled for a school to the Hamilton walk. I then mentioned that they don't hold the record for the furthest distance travelled by a person because that of course belongs to the New Yorker in 1993.

People setting off from Dunsink Observatory on the 2009 walk

There was bit of drama concerning the plaque this year. On the Sunday before the walk I received a phone call from Liam O'Neill of the Cabra Community Council. He informed me that he had just passed the bridge and noticed that the plaque had disappeared. I wondered had the Hamilton story become so famous that the plaque was stolen to be sold to the highest bidder on the black market! Feeling like Sherlock Holmes in the "Strange case of the missing mathematical plaque", I pursued the case on the following day and found out that the City Council had removed the plaque for restoration and were planning to install it in a different location higher up on the bridge. I made it clear that of all the days in the year October 16 was the one day when the plaque needed to be on the bridge. Fortunately, they did their work quickly and had the restored plaque back up in time for the walk on October 16. I was given the honour of unveiling the restored plaque in its new location on the bridge at the end of the walk.

ANTHONY G. O'FARRELL, EFIM ZELMANOV AND FIACRE Ó CAIRBRE ON THE 2009 WALK

3. PART OF THE HAMILTON STORY

William Rowan Hamilton (1805–1865) is Ireland's greatest mathematician and one of the world's most outstanding mathematicians and scientists ever. Born in Dominick Street, Dublin he spent his early youth on the Banks of the Boyne in Trim across from the spectacular ruins of Trim Castle. He then lived in Dunsink Observatory for the rest of his life. See [2], [3], [6] for more on Hamilton's life and works. Like many great mathematicians, Hamilton's motivation for doing mathematics was the quest for beauty. The beauty in mathematics typically lies in the beauty of ideas because mathematics essentially comprises an abundance of ideas. Hamilton was successful in finding much beauty in mathematics. He was also a poet and won the Chancellor's Poetry Prize twice in Trinity College and published his literary work in journals and magazines. He once wrote:

Mathematics is an aesthetic creation, akin to poetry, with its own mysteries and moments of profound revelation.

As is frequently the case in mathematics, practical power was an offspring of his quest for beauty. See [5] where I make a case for why beauty is arguably the most important feature of mathematics.

Hamilton's mathematics has been, and still is, incredibly powerful when applied to science, engineering and many other areas. I will now mention various applications including recent ones like computer games and special effects in movies.

(i) Quaternions play a significant role in computer games. One example of this, which always appeals to journalists, radio hosts and students of course, is the fact that Lara Croft in Tombraider was created using quaternions!

(ii) Continuing with the theme of entertainment, quaternions now play a prominent role in special effects in movies. For example, an Irish company called Havok used quaternions in the creation of the acclaimed new special effects in the movie, The Matrix Reloaded, and also in the movie, Poseidon, which was nominated for an Oscar for its visual effects in 2007. Havok won an Emmy award in the US in 2008 for pioneering new levels of realism and interactivity in movies and games. Also, the dramatic visual effects in the recent James Bond movie, Quantum of Solace, were created by Havok. Students always give a positive reaction when I show a movie character like Shrek etc. and then tell them its creation depends heavily on mathematics.

(iii) Quaternions played a role in Maxwell's mathematical theory and prediction of electromagnetic waves in 1864. Maxwell's theory ultimately led to the detection of radio waves by Hertz. Hence, the inventions of radio, television, radar, X–rays and countless other significant products of our time are directly related to Hamilton's mathematics. Maxwell's work illustrates the "magical" power of mathematics because his mathematics made the invisible visible since radio waves are invisible to our five senses. Maybe mathematics has this "magical" power because it comprises many ideas which are not limited to our five senses.

(iv) Hamilton's new theory of dynamics in 1834 was indispensable for the creation of Quantum Mechanics in the early 1900s. Quantum Mechanics replaced Newtonian Mechanics for helping us understand the physical world at the microscopic level. Also, his famous "Hamiltonian" function is fundamental to many aspects of Physics.

(v) Vector analysis, which is indispensable in Physics, is an offspring of Hamilton's theory of quaternions.

Broombridge has become a world famous site in the history of mathematics and science because of Hamilton's creation of quaternions. The word "Broomsday" is now sometimes used in mathematical circles to indicate October 16 and the walk, and the word plays the same role as "Bloomsday" for literary groups. One may notice that there are different spellings of what I call Broombridge, i.e. Brougham Bridge, Broome Bridge etc. I use Broombridge because that seems to be the current spelling.

Eamon de Valera was a mathematician himself and greatly admired Hamilton. De Valera once declared in Dáil Éireann:

This is the country of Hamilton, a country of great mathematicians.

De Valera lectured in mathematics in Maynooth in 1913–1914. He paid homage to Hamilton with a little graffiti of his own by scratching the quaternion formulas on the wall of his prison cell in Kilmainham jail. Our current Taoiseach should also be familiar with Hamilton, as he observes his image frequently. There is a statue of Hamilton on the steps of Government Buildings in Merrion Street, dating back to when the buildings housed the College of Science. On November 13, 1958 the Taoiseach, Eamon De Valera, unveiled a plaque at Broombridge to commemorate Hamilton's creation of quaternions. The unveiling received substantial coverage in the newspapers the following day. It appeared with a photograph on the front page of the Irish Times and was also prominently featured in the Irish Independent and the Irish Press. The papers quoted De Valera as saying:

I am glad, as head of the Government, to be able to honour the memory of a great scientist and a great Irishman. It is a great personal satisfaction for me to be present, because it was well over fifty years ago since I had heard the story of the bridge and the birth of quaternions. Arthur Conway was professor of mathematical physics at UCD and it was he who had introduced me to the work of Hamilton and told the story of the bridge and how the solution had come to the great mathematician while walking past it. The inspiration came to him in a flash of genius, just as he was about to pass the bridge and with Archimedean exhortation he gave expression to his eureka moment by writing the undying formula on the bridge. On many occasions since I first heard this story I have come to this place as a *holy place. I have searched stone after stone in the hope of finding some trace of that famous inspiration. I did not know until comparatively recent times that Hamilton himself had sought the inscription some fifteen or so years after he had written it and had failed to find it. Time had done its work but we are gathered at the bridge to see that the inscription would be perpetuated and those who passed it would remember that they were passing a spot that was famous in the history of science.*

De Valera then thanked the Dublin Institute for Advanced Studies for erecting the plaque and commended Padraig De Brún and Felix Hackett for doing more than anyone in bringing the project to its conclusion. Descendants of Hamilton, Lady Rowan Hamilton and Hans Rowan Hamilton, were present. Michael Biggs designed the plaque.

In the bicentenary year 2005 we were planning to propose that the bridge at Broombridge be renamed Hamilton Bridge. To our pleasant surprise we found out the little known fact that the name of the bridge had been officially changed to Hamilton Bridge in 1958 (even though there is no physical evidence on the bridge itself). In 2004 Pat Liddy sent me an e-mail from Brigid Johnston of Waterways Ireland, in which she says she found a copy of a letter from the Dublin Institute of Advanced Studies, dated May 21, 1958. In the letter the Registrar of the Institute informed the Chairman of CIE that the "appropriate authorities of Dublin Corporation" had approved the renaming of Broome Bridge in honour of Hamilton. Brigid went on to say that on May 27, 1958 the Deputy Chief Engineer (Civil) at Westland Row wrote to the Assistant Engineer (Canals) at James' St. Harbour to inform him that the name was changed from Broome Bridge to Hamilton Bridge, and the Assistant Engineeer (Canals) passed this information on to his staff the following day. There are also two housing estates, one in Cabra and one in Trim, named after Hamilton.

Of course, Hamilton is not the only person to make a famous link between the Royal Canal and a mathematical concept. Brendan Behan also did with his famous lines!

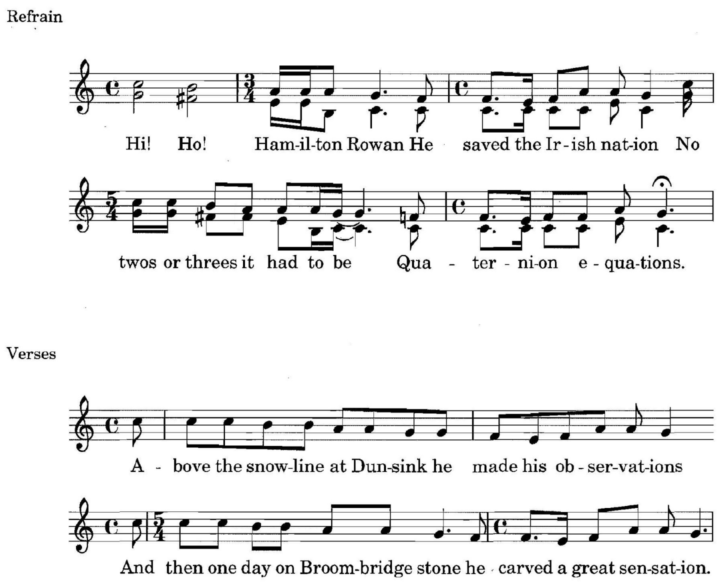
the oul' triangle
goes jingle jangle
along the Banks of the Royal Canal.

4. The Ballad of Rowan Hamilton

Inspired by the Hamilton walk, Jack Gannon wrote this song in 2003 and the sheet music for the song was first published in [1].

The Ballad of Rowan Hamilton

Jack Gannon



Many were the grateful ones, They thought it was a pity
That oonsigh, eejits and the rest, were foremost in the city.

So praise the gallant scientists who favour rhyme and reason
At Broombridge you can see it there, it always is in season.

Music engraving by LilyPond 2.6.5—www.lilypond.org

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BOOK REVIEWS

Mathematical Analysis and Proof

by David S. G. Stirling

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It is clear for every undergraduate Mathematics lecturer in Ireland and the UK that probably the most difficult issue that the Mathematics students have is the idea of rigorous proof. Often wise students regard proofs as unnecessary "why bother to prove a general statement about n when it's clear that it is verified for all the values of n that I have considered") and always fear them. In this book David Stirling uses a mathematical language accessible to the beginner undergraduate student in order to demonstrate that formal proof is a vital part of Mathematics. This is not done just by giving accessible proofs, but rather by showing how they are constructed. Although this book is about Mathematical Analysis, the philosophy of the proof that the author presents is valid for all areas of Mathematics.

In developing mathematical arguments, the author is particularly careful that all the logical connections are clearly presented and emphasizes the way in which the results are deduced. The aspect of good mathematical writing is not forgotten either, some of the sample proofs being real models of elegance. The presentation reads naturally and the material is stimulating and thought provoking.

The text starts with setting the scene and then progresses with notions on logic and its use in deduction, before it launches into subjects like sets and numbers, mathematical induction, inequalities, limits, infinite series, continuity, differentiation, integration and functions of several variables. Every chapter finishes with a set of exercises for some of which hints and solutions are given.

It is my conviction that this book will be of a real help both for the undergraduate students but also for the lecturer who tries to find a suitable way to introduce the students to the intriguing world of mathematical proofs.