MacCool's Proof of Morley's Miracle

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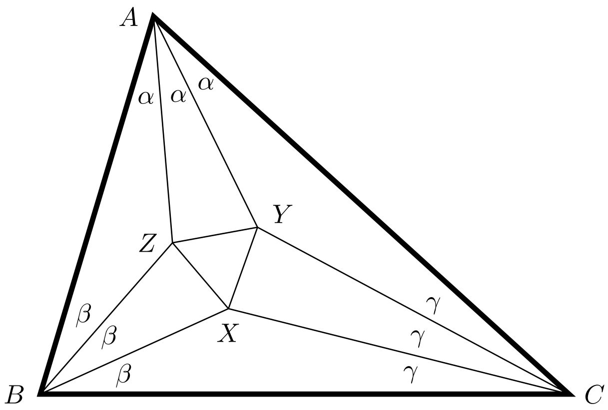
One of the most beautiful results in plane geometry is known as *Morley's Miracle* (1899). In essence it states that the triangle XYZ in the figure below is always equilateral. It features prominently on the front cover of the popular work [2] but is "still not as well-known as it deserves to be" [3]. The excellent web article [1] continues to track its development and also hosts a wide variety of proofs. None of the early proofs was easy but since 1990 elementary ones have emerged which are *backward* in the sense that they start from the equilateral triangle and eventually reconstruct the original. Finding a *direct* proof that matches them in brevity and simplicity has always been an elusive goal [3].



So I was amazed to find just such a proof in MacCool's notebooks and indeed it was so short that I nearly missed it. At first glance he seemed to be merely doodling, but moments later he had finished the proof and was working on something completely different.

Those readers who haven't heard of MacCool's notebooks may be surprised to learn that I am still less than halfway through the first one. Translation from the Ogham script is proving a long slow process and I am deeply indebted to one correspondent who reviewed and improved upon my original efforts, often spotting intricacies that I had overlooked. Although the gist of his arguments is always clear MacCool delights in recording only a minimum of information, and this particular proof was little more than a sketch decorated with jottings of line segments and angles. Like all the rest so far it is based solely on straight line geometry and similar triangles, but anyone interested in more advanced concepts may be pleased to know that diagrams containing circles begin to appear early in book two.

In his doodle the unit of measure is the perpendicular DX, and the lengths of BX and CX are s and s'. E and F are points on BC where $\angle BXE = \angle FXC = 60^o$. P and P' are where $BP = s$ and $CP' = s'$ and S is constructed so that $BS = s$ and $\angle SBX = 120^o$. This makes the four marked angles 60^o (even if ΔABC is obtuse). The rest of his construction is self-explanatory.
Now by (vi) and (vii) $2ST = 2SU + 2UT = s + 2\left(s - 2s^{-1}\right) = 3s - 4s^{-1}$ and $\Delta BQV \sim \Delta BDX$ yields $VQ = 1 - 4s^{-2}$ so $PQ = PV + VQ = 3 - 4s^{-2}$ thus

$$2ST = sPQ.$$

Then by (iv) and (v)

$$AY = \left(\frac{AC}{s'}\right)\left(\frac{PS}{ST}\right) = \frac{2AC.PS}{ss'PQ}.$$

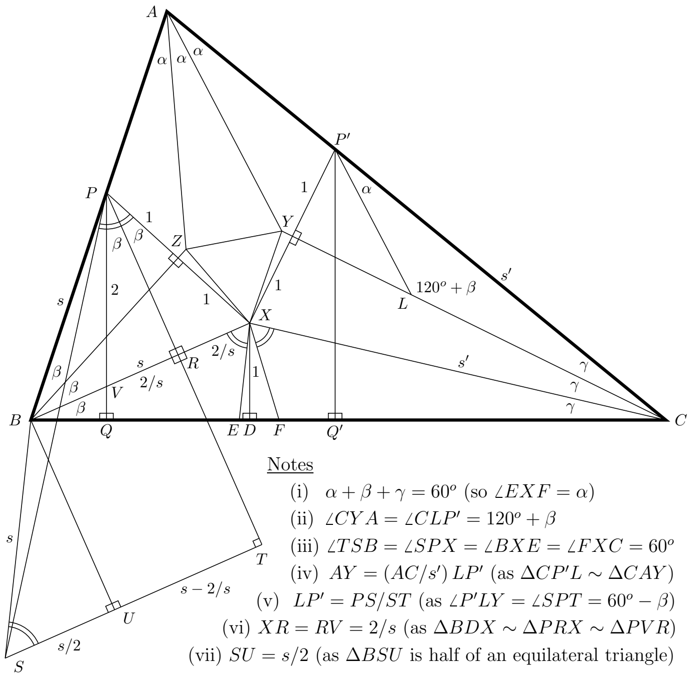
If W is the midpoint of PS then since ΔBSP is isosceles ΔBWP and ΔEDX will have identical angles, hence $\Delta BWP \sim \Delta EDX$ giving $XE = 2s/PS$. Therefore

$$XE.AY = \frac{4AC}{s'PQ}$$

and, by symmetry,

$$\frac{XE.AY}{XF.AZ} = \left(\frac{4AC}{s'PQ}\right)\left(\frac{sP'Q'}{4AB}\right) = \frac{sAC.P'Q'}{s'AB.PQ} = 1$$

because $PQ(AB/s) = P'Q'(AC/s')$ is the height of ΔABC. However this means $AZ : AY = XE : XF$ and as $\angle ZAY = \alpha = \angle EXF$



then $\Delta AZY \sim \Delta XEF$. Hence $\angle YZA = \angle FEX = 60^o + \beta$ and $\angle AYZ = \angle XFE = 60^o + \gamma$. Analogous arguments for ΔBXZ and ΔCYX show $\angle ZXB = 60^o + \gamma$, $\angle CXY = 60^o + \beta$ and $\angle BZX = \angle CYX = 60^o + \alpha$. All the angles in the doodle may now be deduced in terms of α, β, γ and it transpires that every angle of ΔXYZ is 60^o.

Here are some comments on the proof leading to a slight variation that may help to make it more intuitive. The underlying idea is to treat it as a series of left/right linkages. The results that $2ST = sPQ$ and $XE = 2s/PS$ are clearly "internal" to the left hand side. On the other hand LP' has a foot in each camp since it can be expressed both in terms of objects from the left PS/ST and objects from the right $s'AY/AC$. Equating these expressions gives a "cross-linkage" $XE.PQ = 4AC/(s'AY)$ and its companion $4AB/(sAZ) = XF.P'Q'$ which may then be combined to form the complicated looking quotient above. Even MacCool seems to have been shocked by the final devastating cross-linkage $PQ.AB/s = P'Q'.AC/s'$ which reduces this quotient to unity. After that the rest is plain sailing.

References

[1] A. Bogomolny, http://www.cut-the-knot.org/triangle/Morley/index.shtml Morley's Miracle *from Interactive Mathematics Miscellany and Puzzles*

[2] H. S. M. Coxeter, and S. L. Greitzer, *Geometry Revisited*, Mathematical Association of America, 1967.

[3] R. K. Guy, The Lighthouse Theorem, Morley and Malfatti — A budget of paradoxes, *Amer. Math. Monthly*, **114** (2007), 97–141.

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Received on 20 March 2009 and in revised form on 11 May 2009.