Irish Math. Soc. Bulletin **62** *(2008), 37–41*

Hermitian Morita Theory: a Matrix Approach

DAVID W. LEWIS AND THOMAS UNGER

Abstract. In this note an explicit matrix description of hermitian Morita theory is presented.

1. Introduction

Let K be a field of characteristic different from two and let A be a central simple K-algebra equipped with an involution $*$. By a well-known theorem of Wedderburn, A is of the form $M_n(D)$, a full matrix algebra over a division K-algebra D. Furthermore, there exists an involution— on D of the same kind as $*$ such that $*$ and—have the same restriction to K. Then $*$ is the adjoint involution ad_{h_0} of some nonsingular ε_0-hermitian form h_0 over $(D, -)$,

$$h_0 : D^n \times D^n \longrightarrow D,$$

with $\varepsilon_0 = \pm 1$. Thus

$$X^* = \mathrm{ad}_{h_0}(X) = S\overline{X}^t S^{-1}, \quad \forall X \in M_n(D),$$

where $S \in GL_n(D)$ is the matrix of h_0, so that $\overline{S}^t = \varepsilon_0 S$.

Let $\mathrm{Gr}_\varepsilon(A, *)$ and $W_\varepsilon(A, *)$ denote the Grothendieck group and Witt group of ε-hermitian forms over $(A, *)$, respectively. Hermitian Morita theory furnishes us with isomorphisms

$$\mathrm{Gr}_\varepsilon(A, *) \cong \mathrm{Gr}_{\varepsilon_0\varepsilon}(D, -) \quad \text{and} \quad W_\varepsilon(A, *) \cong W_{\varepsilon_0\varepsilon}(D, -).$$

These isomorphisms are the result of the following equivalences of categories

$$\left\{\begin{matrix}\varepsilon\text{-hermitian}\\ \text{forms over}\\ (M_n(D), *)\end{matrix}\right\} \xleftrightarrow{\text{scaling}} \left\{\begin{matrix}\varepsilon_0\varepsilon\text{-hermitian}\\ \text{forms over}\\ (M_n(D), -^t)\end{matrix}\right\} \xleftrightarrow[\text{equivalence}]{\text{Morita}} \left\{\begin{matrix}\varepsilon_0\varepsilon\text{-hermitian}\\ \text{forms over}\\ (D, -)\end{matrix}\right\}$$

(all forms are assumed to be nonsingular) which respect isometries, orthogonal sums and hyperbolic forms.

In this note we describe these correspondences explicitly. In particular we give a matrix description of Morita equivalence which does not seem to be generally known. Other explicit descriptions can be found in [3, 4, 5]. The subject is often treated in a more abstract manner, such as in [1] and [2, Chap. I, §9].

2. Scaling

Let M be a right $M_n(D)$-module and let $h : M \times M \longrightarrow M_n(D)$ be an ε-hermitian form with respect to $*$, i.e.

$$h(y,x) = \varepsilon h(x,y)^* = \varepsilon S \overline{h(x,y)}^t S^{-1}.$$

Proposition 2.1. *The form*

$$S^{-1}h : M \times M \longrightarrow M_n(D),\ (x,y) \longmapsto S^{-1}h(x,y)$$

is $\varepsilon_0\varepsilon$-hermitian over $(M_n(D), \overline{}^t)$.

Proof. Sesquilinearity of $S^{-1}h$ with respect to $\overline{}^t$ follows easily from sesquilinearity of h with respect to $*$:

$$\begin{aligned}(S^{-1}h)(x\alpha,y) &= S^{-1}h(x\alpha,y) = S^{-1}\alpha^* h(x,y)\\ &= S^{-1}S\overline{\alpha}^t S^{-1}h(x,y) = \overline{\alpha}^t S^{-1}h(x,y)\end{aligned}$$

for any $\alpha \in M_n(D)$ and any $x, y \in M$.

Furthermore, using the fact that $\overline{S}^t = \varepsilon_0 S$, we get

$$\begin{aligned}(S^{-1}h)(y,x) &= S^{-1}h(y,x)\\ &= S^{-1}\varepsilon S\overline{h(x,y)}^t S^{-1}\\ &= \varepsilon\overline{h(x,y)}^t S^{-1}\\ &= \varepsilon\varepsilon_0\overline{h(x,y)}^t\overline{(S^{-1})}^t\\ &= \varepsilon\varepsilon_0\overline{(S^{-1}h)(x,y)}^t\end{aligned}$$

for any $x, y \in M$. ∎

Remark 2.2. By the first part of the proof, scaling of a sesquilinear form h (rather than an ε-hermitian form h) with respect to $*$ results in a sesquilinear form $S^{-1}h$ with respect to $\overline{}^t$.

Remark 2.3. The matrix S is not determined uniquely, but only up to scalar multiplication by $\lambda \in K$, since λS and S give the same involution ad_{h_0}. Hence the scaling correspondence is not canonical.

3. MORITA EQUIVALENCE

Every module over $M_n(D) \cong \mathrm{End}_D(D^n)$ is a direct sum of simple modules, namely copies of D^n. Let $(D^n)^k$ be such a module. We identify $(D^n)^k$ with $D^{k\times n}$, the $k \times n$-matrices over D. We view each row of a $k \times n$-matrix over D as an element of D^n. Note that $M_n(D)$ acts on $D^{k\times n}$ on the right.

Now let

$$h : D^{k\times n} \times D^{k\times n} \longrightarrow M_n(D)$$

be an ε-hermitian form over $(M_n(D), -^t)$.

Proposition 3.1. *There exists an ε-hermitian $k \times k$-matrix $B \in M_k(D)$ such that*

$$h(x, y) = \overline{x}^t By, \ \forall x, y \in D^{k\times n}. \tag{1}$$

Proof. Let $B = (b_{ij})$. We will determine the entries b_{ij}. Let $e_{ij} \in D^{k\times n}$, $e'_{ij} \in D^{n\times k}$ and $E_{ij} \in M_n(D)$ respectively denote the $k \times n$-matrix, the $n \times k$-matrix and the $n \times n$-matrix with 1 in the (i, j)-th position and zeroes everywhere else. One can easily verify that

$$e_{if} E_{f\ell} = e_{i\ell}, \tag{2}$$

where $1 \le i \le k$ and $1 \le f, \ell \le n$. Also note that if $C \in M_n(D)$, then computing the product $E_{ij}C$ picks the j-th row of C and puts it in row i while making all other entries zero. Similarly, computing the product CE_{ij} picks the i-th column of C and puts it in column j while making all other entries zero. The matrices e_{ij} and e'_{ij} behave in a similar fashion.

The matrices $\{e_{ij} \mid 1 \le i \le k, \ 1 \le j \le n\}$ generate $D^{k\times n}$ as a right $M_n(D)$-module. Thus it suffices to compute $h(e_{if}, e_{jg})$ where $1 \le i, j \le k$ and $1 \le f, g \le n$. Let us first compute $h(e_{ii}, e_{jj})$:

$$\begin{aligned} h(e_{ii}, e_{jj}) &= h(e_{ii}E_{ii}, e_{jj}E_{jj}) \\ &= E_{ii} h(e_{ii}, e_{jj}) E_{jj} \\ &= m_{ij} E_{ij}, \end{aligned}$$

where m_{ij} is the (i, j)-th entry of $h(e_{ii}, e_{jj}) \in M_n(D)$. In other words, the matrix $h(e_{ii}, e_{jj})$ has only one non-zero entry, namely m_{ij} in position (i, j).

Next, let us compute $h(e_{if}, e_{jg})$. We will use the fact that

$$e_{if} = e_{ii} E_{if},$$

where $1 \leq i \leq k$ and $1 \leq f \leq n$, which follows from (2). We get

$$\begin{aligned} h(e_{if}, e_{jg}) &= h(e_{ii} E_{if}, e_{jj} E_{jg}) \\ &= E_{fi} h(e_{ii}, e_{jj}) E_{jg} \\ &= \big(h(e_{ii}, e_{jj})\big)_{ij} E_{fg} \\ &= m_{ij} E_{fg}. \end{aligned}$$

Let $b_{ij} = m_{ij}$ where $1 \leq i, j \leq k$. We have

$$\begin{aligned} \overline{e_{if}}^t B e_{jg} &= e'_{fi} B e_{jg} \\ &= b_{ij} E_{fg} \\ &= m_{ij} E_{fg}. \end{aligned}$$

Therefore, $h(e_{if}, e_{jg}) = \overline{e_{if}}^t B e_{jg}$ where $1 \leq i, j \leq k$ and $1 \leq f, g \leq n$, which establishes (1).

Finally,

$$m_{ji} E_{ji} = h(e_{jj}, e_{ii}) = \varepsilon \overline{h(e_{ii}, e_{jj})}^t = \varepsilon \overline{m_{ij}} E_{ji}, \text{ for } 1 \leq i, j \leq k,$$

which implies $m_{ji} = \varepsilon \overline{m_{ij}}$, for $1 \leq i, j \leq k$. In other words, $\overline{m_{ji}} = \varepsilon m_{ij}$, for $1 \leq i, j \leq k$, so that $\overline{B}^t = \varepsilon B$, which finishes the proof. ∎

So, given an ε-hermitian form h over $(M_n(D), -^t)$, we have obtained an ε-hermitian form over $(D, -)$ with matrix B as in Proposition 3.1. Conversely, given an ε-hermitian form

$$\varphi : D^k \times D^k \longrightarrow D,$$

represented by the matrix B (i.e., $B = \big(\varphi(e_i, e_j)\big)$ for a D-basis $\{e_i\}$ of D^k), we define

$$h : D^{k \times n} \times D^{k \times n} \longrightarrow M_n(D)$$

by

$$h(x, y) := \overline{x}^t B y, \ \forall x, y \in D^{k \times n},$$

which gives an ε-hermitian form over $(M_n(D), -^t)$.

Remark 3.2. The correspondence $h \leftrightarrow \varphi$ already works for forms that are just sesquilinear, without assuming any hermitian symmetry. Since scaling also preserves sesquilinearity, as remarked earlier, we conclude that the category equivalences of §1 already hold for sesquilinear forms over $(M_n(D), *)$, $(M_n(D), -^t)$ and $(D, -)$, respectively.

References

[1] A. Fröhlich and A. M. McEvett, Forms over rings with involution, *J. Algebra* **12** (1969), 79–104.

[2] M.-A. Knus, *Quadratic and Hermitian forms over rings*, Grundlehren der Mathematischen Wissenschaften **294**, Springer-Verlag, Berlin, 1991.

[3] D. W. Lewis, Forms over real algebras and the multisignature of a manifold, *Advances in Math.* **23** (1977), no. 3, 272–284.

[4] C. Riehm, Effective equivalence of orthogonal representations of finite groups, *J. Algebra* **196** (1997), no. 1, 196–210.

[5] C. Riehm, Orthogonal, symplectic and unitary representations of finite groups, *Trans. Amer. Math. Soc.* **353** (2001), no. 12, 4687–4727.

David Lewis and Thomas Unger,
School of Mathematical Sciences,
University College Dublin,
Belfield, Dublin 4, Ireland,
david.lewis@ucd.ie, thomas.unger@ucd.ie

Received on 19 December 2008.