Optimisation Problems for the Determinant of a Sum of 3×3 Matrices

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Abstract. Given a pair of positive definite 3×3 matrices A, B, the maximum and minimum values of $\det(U^*AU + V^*BV)$ are determined when U, V vary within the collection of unitary 3×3 matrices.

1. Introduction

Let m, n be a pair of natural numbers. Suppose $A_1, A_2, \ldots, A_n$ are $m \times m$ Hermitian positive definite matrices. What are the maximum and minimum values of the expression

$$\det\Big(\sum_{i=1}^{n} U_i^* A_i U_i\Big)$$

as $U_1, U_2, \ldots, U_n$ range over the group G_m of $m \times m$ unitary matrices? The case $m = 2$ of this arose in the context of an interesting maximum-likelihood problem which is discussed in [3], and the minimum value was determined there when the given matrices were real and symmetric, and the Us members of the subgroup of G_2 of orthogonal matrices.

In this note we address the above problem only in the case $m = 3$, and resolve it when $n = 2$. However, the methods used here don't appear to generalise to the case of general m, even when $n = 2$. Accordingly, a different strategy has been devised to deal with this more general case, which will be the subject of another paper. However, at the time of writing, the general case of arbitrary m, n remains open.

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2. Statement of the Main Result

Theorem 1. *Let S and T be two 3×3 positive definite matrices with spectra $\sigma(S) = \{s_1, s_2, s_3\}$ and $\sigma(T) = \{t_1, t_2, t_3\}$, respectively, where $s_1 \geq s_2 \geq s_3 > 0$ and $t_1 \geq t_2 \geq t_3 > 0$. Then*

$$\min\{\det(S + U^*TU) : U \in G_3\} = \prod_{i=1}^{3}(s_i + t_i),$$

and

$$\max\{\det(S + U^*TU) : U \in G_3\} = \prod_{i=1}^{3}(s_i + t_{4-i}).$$

3. Two Preparatory Lemmas

Lemma 1. *Let $A = [a_{ij}]$ be a 3×3 matrix. Let*

$$M = \begin{bmatrix} x + a_{11} & a_{12} & a_{13} \\ a_{21} & y + a_{22} & a_{23} \\ a_{31} & a_{32} & z + a_{33} \end{bmatrix}.$$

Then
$\det M = xyz + yza_{11} + zxa_{22} + xya_{33} + xA_{11} + yA_{22} + zA_{33} + \det A.$

Proof. Here and later, we use the customary notation A_{ij} for the cofactor of the typical element a_{ij}, so that, in particular, A_{11}, A_{22}, A_{33} are the principal minors of A of order 2×2. Expanding by elements of the first row,

$$\begin{aligned}
\det M &= (x + a_{11})[(y + a_{22})(z + a_{33}) - a_{23}a_{32}] \\
&\quad - a_{12}[a_{21}(z + a_{33}) - a_{31}a_{23}] + a_{13}[a_{21}a_{32} - a_{31}(y + a_{22})] \\
&= (x + a_{11})(y + a_{22})(z + a_{33}) - [xa_{23}a_{32} + ya_{13}a_{31} + za_{12}a_{21}] \\
&\quad - a_{11}a_{23}a_{32} - a_{12}[a_{21}a_{33} - a_{31}a_{23}] + a_{13}[a_{21}a_{32} - a_{31}a_{22}] \\
&= (x + a_{11})(y + a_{22})(z + a_{33}) - a_{11}a_{23}a_{32} \\
&\quad - [xa_{23}a_{32} + ya_{13}a_{31} + za_{12}a_{21}] - a_{12}A_{12} + a_{13}A_{13} \\
&= xyz + xya_{33} + yza_{22} + zxa_{11} + x[a_{22}a_{33} - a_{23}a_{32}] \\
&\quad + y[a_{11}a_{33} - a_{13}a_{31}] + z[a_{11}a_{22} - a_{12}a_{21}] \\
&\qquad + a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13} \\
&= xyz + xya_{33} + yza_{22} + zxa_{11} + xA_{11} + yA_{22} + zA_{33} + \det A.
\end{aligned}$$

□

We wish to exploit this result when $A = U^*TU$, where T is a diagonal matrix with positive diagonal elements $p \geq q \geq r > 0$, and $U = [u_{ij}]$ is unitary. A calculation shows that

$$a_{ij} = p\overline{u_{i1}}u_{j1} + q\overline{u_{i2}}u_{j2} + r\overline{u_{i3}}u_{j3},\ i,j = 1,2,3.$$

In particular,

$$a_{ii} = p|u_{i1}|^2 + q|u_{i2}|^2 + r|u_{i3}|^2,\ i = 1,2,3.$$

In addition, A is invertible and $A^{-1} = U^*T^{-1}U = (\det A)^{-1}[A_{ij}]^t$, whence

$$\frac{A_{ii}}{pqr} = p^{-1}|u_{i1}|^2 + q^{-1}|u_{i2}|^2 + r^{-1}|u_{i3}|^2,\ i = 1,2,3,$$

or

$$A_{ii} = qr|u_{i1}|^2 + rp|u_{i2}|^2 + pq|u_{i3}|^2,\ i = 1,2,3.$$

Observe too that

$$\sum_{i=1}^{3} |u_{ij}|^2 = \sum_{j=1}^{3} |u_{ij}|^2 = 1,\ i,j = 1,2,3,$$

and so the matrix $[|u_{ij}|^2]$ is doubly-stochastic. With this in mind we prove a rearrangement inequality.

Lemma 2. *Let $[p_{ij}]$ stand for an arbitrary $n \times n$ doubly-stochastic matrix. Let a, b be two real $n \times 1$ vectors whose entries are in decreasing order. Then*

$$\sum_{i=1}^{n} a_i b_{n-i+1} \leq \sum_{i,j=1}^{n} a_i b_j p_{ij} \leq \sum_{i=1}^{n} a_i b_i.$$

Proof. Consider the function f defined on the convex set $\mathcal{P}$ of all $n \times n$ doubly-stochastic matrices $P = [p_{ij}]$ by

$$f(P) = \sum_{i,j=1}^{n} a_i b_j p_{ij},\quad P \in \mathcal{P}.$$

Clearly, f is linear in P, and so convex on $\mathcal{P}$. Hence it attains its maximum and minimum at an extreme point of $\mathcal{P}$. But, by Birkhoff's theorem [1], the set of extreme points of the latter consists of the set of permutation matrices $\{\pi(I) = [\delta_{i\pi(j)}] : \pi \in S_n\}$, where S_n

denotes the group of permutations of $\{1, 2, \dots, n\}$. Hence

$$\begin{aligned}
\min\{f(P) : P \in \mathcal{P}\} &= \min\{f(\pi(I)) : \pi \in S_n\} \\
&= \min\{\sum_{i,j=1}^{n} a_i b_j \delta_{i\pi(j)} : \pi \in S_n\} \\
&= \min\{\sum_{j=1}^{n} a_{\pi(j)} b_j : \pi \in S_n\} \\
&= \sum_{j=1}^{n} a_j b_{n-j+1},
\end{aligned}$$

by the elementary rearrangement inequality, since a, b are similarly ordered [2]. This argument establishes that

$$\sum_{i=1}^{n} a_i b_{n-i+1} \leq \sum_{i,j=1}^{n} a_i b_j p_{ij},$$

with equality when $p_{ij} = \delta_{i(n-j+1)},\ i, j = 1, 2, \dots, n,$.

The maximum can be handled in the same way. □

4. Proof of the Main Result

Define F on the group G_3 of 3×3 unitary matrices by

$$F(U) = \det(S + U^*TU),\ U \in G_3.$$

In the first place, there are matrices $V, W \in G_3$ such that

$$S = V \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} V^* \equiv V \Delta V^*,$$

and

$$T = W \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix} W^* \equiv W \Lambda W^*,$$

say. Hence

$$F(WUV^*) = \det(\Delta + U^* \Lambda U),$$

whence it's enough to deal with the case where $S = \Delta, T = \Lambda$. This being so, we can appeal to Lemma 1, taking

$$A = U^* \Delta U = \Big[\sum_{k=1}^{3} t_k \overline{u_{ik}} u_{jk}\Big],$$

and obtain that

$$\begin{aligned}
\det(\Delta + U^*\Lambda U) &= \det(\Delta + A) \\
&= s_1 s_2 s_3 + s_1 s_2 s_3 \sum_{k=1}^{3} s_k^{-1} a_{kk} + \sum_{k=1}^{3} s_k A_{kk} + \det A \\
&= s_1 s_2 s_3 + s_1 s_2 s_3 \sum_{i=1}^{3} s_i^{-1} \sum_{j=1}^{3} t_j |u_{ij}|^2 \\
&\qquad + t_1 t_2 t_3 \sum_{i=1}^{3} s_i \sum_{j=1}^{3} t_j^{-1} |u_{ij}|^2 + t_1 t_2 t_3 \\
&= s_1 s_2 s_3 + s_1 s_2 s_3 \sum_{i,j=1}^{3} s_i^{-1} t_j |u_{ij}|^2 \\
&\qquad + t_1 t_2 t_3 \sum_{i,j=1}^{3} s_i t_j^{-1} |u_{ij}|^2 + t_1 t_2 t_3 \\
&\geq s_1 s_2 s_3 + s_1 s_2 s_3 \sum_{i=1}^{3} s_i^{-1} t_i + t_1 t_2 t_3 \sum_{i=1}^{3} s_i t_i^{-1} + t_1 t_2 t_3,
\end{aligned}$$

by Lemma 2, since s_1, s_2, s_3, and $t_1^{-1}, t_2^{-1}, t_3^{-1}$ are oppositely ordered. It follows that

$$\begin{aligned}
\det(\Delta + U^*\Lambda U) &\geq s_1 s_2 s_3 + t_1 s_2 s_3 + t_2 s_1 s_3 + t_3 s_1 s_2 \\
&\qquad + s_1 t_2 t_2 + t_2 s_1 s_3 + t_3 s_1 s_2 + t_1 t_2 t_3 \\
&= (s_1 + t_1)(s_2 + t_2)(s_3 + t_3),
\end{aligned}$$

with equality when $U = I$, the identity matrix. Hence

$$\min\{F(U) : U \in G_3\} = \prod_{i=1}^{3} (s_i + t_i).$$

Arguing in a similar manner, it can be seen that

$$\max\{F(U) : U \in G_3\} = \prod_{i=1}^{3} (s_i + t_{4-i}).$$

This completes the proof of Theorem 1.

References

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