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Lipschitz Character of Solutions to the Inner Obstacle Problems

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Abstract. In our paper we consider the inner problem with $l \in \mathbb{N}$ impediments from below, the inner problem with $m \in \mathbb{N}$ impediments from above and the double inner problem with $l + m$ impediments. Assuming the Lipschitz character of the obstacles we show that the corresponding solutions are also Lipschitz. We extend here the result given in [SV], where the author considered the inner obstacle problem with a single impediment from below. Our work is based on the ideas introduced by J. Jordanov from 1982 who investigated $H^{1,p}(\Omega)$ regularity of solutions to inner obstacle problems.

1. Introduction

In many physical processes "obstacles" appear in a natural way having strong influence on the character of the examined problem. A simple example of such a situation is the study of contrast between a vibrating membrane and a vibrating membrane set between obstacles.

In the 1970's there was considerable interest in the analysis of obstacle problems. This was connected with the development of research on variational inequalities and has been studied by many authors (see [BC], [BS], [T] and references therein). The majority of results concentrated on, natural from a mathematical point of view, problems of existence and uniqueness of the solutions. However, in case of variational inequalities corresponding to obstacle problems additional questions regarding, e.g., the coincidence set (cf. [DS1], [DS2]) or regularity of the solutions (cf. [BS]) can be posed. These problems seem to be interesting due to possible applications.

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The fundamental result where regularity of solutions with regard to regularity of obstacles is studied in case of global obstacle problems can be found in [KS].

Recently the interest in the analysis of the obstacle problems has increased. This is due to appearance of works on the inner problems (see [BSz], [JOS], [Ro] and references therein). Among other things examination of regularity of solutions to the inner obstacle problems is a matter of significant importance.

A study of the Lipschitz character of the solutions to the obstacle problems was initiated in [SV]. The authors showed that the solution of the global problem and the inner problem with one obstacle from below is Lipschitz continuous assuming that the impediments are Lipschitz. Later on the papers [Ch1], [Ch2] appeared where one can find theorems concerning the Lipschitz continuity of the solutions to the global inverse and double global problems.

In our work we aim at transferring results concerning Lipschitz character of solutions of global obstacle problems to the case of inner ones. It is worth mentioning that the construction presented in our paper enables us to identify each inner problem with the corresponding global one. This fact makes it possible to carry out the complete analysis of the inner problems with the help of methods available for the global ones.

We offer a comprehensive study of the Lipschitz regularity of solutions of the inner obstacle problems. The present paper is a part of the research program on free boundary problems.

2. Notation and Basic Definitions

Throughout the paper we assume that $\Omega \subset \mathbb{R}^n$ is an open, bounded set with the smooth boundary $\partial\Omega$. The functions $a_{ij} : \bar{\Omega} \to \mathbb{R}$ for $1 \le i, j \le n$ belong to $C^1(\Omega)$ and satisfy the ellipticity condition, i.e., there exist $\gamma, \mu > 0$ such that

$$\mu|\xi|^2 \ge a_{ij}(x)\xi_i\xi_j \ge \gamma|\xi|^2 \qquad \text{for } x \in \Omega \text{ and } \xi \in \mathbb{R}^n, \tag{1}$$

where the summation convention is adopted. We also introduce the second order elliptic operator

$$L = -\partial_{x_i}\big(a_{ij}(x)\partial_{x_j}\big). \tag{2}$$

Remark 2.1. The operator L defined by (2) considered as the mapping $L : H_0^1(\Omega) \to H^{-1}(\Omega)$ defines (see [KS]) a bilinear, continuous and coercive form on $H^1_0(\Omega)$ as follows:

$$a(u,v) = \langle Lu, v\rangle = \int_\Omega a_{ij}(x) u_{x_i}(x) v_{x_j}(x)\, dx \quad (u,v \in H^1_0(\Omega)). \quad (3)$$

Now we pass to the precise definitions of fundamental concepts of this work. Let us consider $l \in \mathbb{N}$ functions $\Psi_i \in H^1(E_i)$ where $E_i \subset \Omega$ are compact sets such that ∂E_i is smooth, $E_i \cap E_j = \emptyset$ for $i, j = 1, \ldots, l$ and $i \neq j$. Next we take $m \in \mathbb{N}$ functions $\Phi_i \in H^1(F_i)$ where $F_i \subset \Omega$ are compact sets such that ∂F_i is smooth, $F_i \cap F_j = \emptyset$ for $i, j = 1, \ldots, m$ and $i \neq j$. Moreover, we assume that:

$$\Psi_i \leq \Phi_j \quad \text{on } E_i \cap F_j \text{ for } i = 1, \ldots, l \text{ and } j = 1, \ldots, m. \quad (4)$$

We denote by K_l, K^m and K_l^m the following admissible sets:

$$\begin{aligned} K_l &= \{v \in H^1_0(\Omega) \;:\; v \geq \Psi_i \text{ on } E_i \text{ for } 1 \leq i \leq l\}, & (5)\\ K^m &= \{v \in H^1_0(\Omega) \;:\; v \leq \Phi_i \text{ on } F_i \text{ for } 1 \leq i \leq m\}, & (6)\\ K_l^m &= \{v \in H^1_0(\Omega) \;:\; v \geq \Psi_i \text{ on } E_i \;\wedge v \leq \Phi_j \text{ on } F_j, & (7)\\ &\qquad \text{for } 1 \leq i \leq l, 1 \leq j \leq m\}. \end{aligned}$$

Definition 2.2. Let l be a fixed natural number. For the form defined by (3) and $f \in H^{-1}(\Omega)$ the problem:
Find $u_l \in K_l$ such that

$$a(u_l, v - u_l) \geq \langle f, v - u_l\rangle \quad \text{for any } v \in K_l, \quad (8)$$

where K_l is defined by (5) is called an $l-$inner obstacle problem with the impediments Ψ_i $(i = 1, \ldots, l)$.

We shall use the notation $l - IP$ to denote the $l-$inner obstacle problem.

Definition 2.3. Let m be a fixed natural number. For the form defined by (3) and $f \in H^{-1}(\Omega)$ the problem:
Find $u^m \in K^m$ such that

$$a(u^m, v - u^m) \geq \langle f, v - u^m\rangle \quad \text{for any } v \in K^m, \quad (9)$$

where K^m is defined by (6) is called an $m-$inner inverse obstacle problem with the impediments Φ_i $(i = 1, \ldots, m)$.

We shall use the notation $m-IIP$ to denote the $m-$inner inverse obstacle problem.

Definition 2.4. Let l, m be fixed natural numbers. For the form defined by (3) and $f \in H^{-1}(\Omega)$ the problem:

Find $u_l^m \in K_l^m$ such that

$$a(u_l^m, v - u_l^m) \geq \langle f, v - u_l^m \rangle \quad \text{for any } v \in K_l^m, \tag{10}$$

where K_l^m is defined by (7) is called an $l, m-$double inner obstacle problem with the impediments Ψ_i $(i = 1, \ldots, l)$ and Φ_j $(j = 1, \ldots, m)$.

We shall use the notation $l, m - DIP$ to denote the $l, m-$double inner obstacle problem.

Remark 2.5. If we put $l = 1$, take $E_1 = \Omega$ and assume that $\Psi_1 = \Psi$ satisfies $\Psi\big|_{\partial\Omega} \leq 0$ then Definition 2.2 in fact is identical with the definition of the global obstacle problem with the impediment Ψ. The admissible set $\tilde{K}_1$ will be defined by

$$\tilde{K}_1 = \{v \in H_0^1(\Omega) \ : \ v \geq \Psi \text{ on } \Omega\}. \tag{11}$$

We shall use the notation GP to denote the global obstacle problem.

Remark 2.6. If we put $m = 1$, take $F_1 = \Omega$ and assume that $\Phi_1 = \Phi$ satisfies $\Phi\big|_{\partial\Omega} \geq 0$ then Definition 2.3 gives the definition of the inverse global obstacle problem with the impediment Φ.The admissible set $\tilde{K}^1$ will be defined by

$$\tilde{K}^1 = \{v \in H_0^1(\Omega) \ : \ v \leq \Phi_i \text{ on } \Omega\}. \tag{12}$$

We shall use the notation GIP to denote the inverse global obstacle problem.

Remark 2.7. If we put $l = m = 1$, take $E_1 = F_1 = \Omega$ and assume that $\Psi_1 = \Psi$ and $\Phi_1 = \Phi$ are such that $\Psi\big|_{\partial\Omega} \leq 0$, $\Phi\big|_{\partial\Omega} \geq 0$ and $\Psi \geq \Phi$ then Definition 2.4 gives the definition of the double global obstacle problem with the impediments Ψ and Φ. The admissible set $\tilde{K}_1^1$ will be defined by

$$\tilde{K}_1^1 = \{v \in H_0^1(\Omega) \ : \ \Psi \leq v \leq \Phi \text{ on } \Omega\}. \tag{13}$$

We shall use the notation DGP to denote the double global obstacle problem.

The existence and uniqueness theorems for GP, GIP, DGP can be found in [KS].

3. Lipschitz Regularity

In this section we present the main results of our paper, i.e., the Lipschitz regularity of the solutions in the case of the inner obstacle problems. Since our approach is based on identification of the inner problem with the corresponding global one, we now recall the following lemma (see [SV], [Ch1], [Ch2]).

Lemma 3.1. *The solutions: $\tilde{u}_1$ — of GP with the impediment Ψ and $f = 0$, $\tilde{u}^1$ — of GIP with the impediment Φ and $f = 0$, $\tilde{u}_1^1$ — of DGP with the impediments Ψ, Φ and $f = 0$ are Lipschitz ($\tilde{u}_1$, $\tilde{u}^1$, $\tilde{u}_1^1 \in H^{1,\infty}(\Omega)$) provided Ψ, $\Phi \in H^{1,\infty}(\Omega)$.*

We start with presenting the result for $l - IP$ (for $l \geq 1$). The following theorem is a generalisation of the one included in [SV] where $l = 1$.

Theorem 3.2. *If $\Psi_i \in H^{1,\infty}(E_i)$ then there exists a unique solution u_l to $l - IP$ with the impediments Ψ_i ($i = 1, \ldots, l$). Moreover, if $f = 0$ this solution is Lipschitz continuous.*

Proof. Let us construct for each $i = 1, \ldots, l$ the functions $\tilde{\Psi}_i : \Omega \to \mathbb{R}$ in the following way

$$\tilde{\Psi}_i = \begin{cases} w_i & \text{in } \Omega \setminus E_i \\ \Psi_i & \text{in } E_i, \end{cases} \tag{14}$$

where $w_i \in H^1(\Omega \setminus E_i)$ solves the following problem

$$\begin{cases} Lw_i = 0 & \text{in } \Omega \setminus E_i \\ w_i = \Psi_i & \text{in } \partial E_i \\ w_i = 0 & \text{in } \partial \Omega. \end{cases} \tag{15}$$

Next we put

$$\Psi = \max\{\tilde{\Psi}_1, \ldots, \tilde{\Psi}_l\}.$$

It is well known (see [KS]) that in order to show existence and uniqueness of the solution u_l of $l - IP$ it is enough to show non-emptiness of the set K_l. It is easy to see that $\Psi \in K_l$ which yields the desired conclusion.

On the other hand, we observe that $\tilde{\Psi}_i$ ($i = 1, \ldots, l$) and consequently Ψ are Lipschitz continuous. Then we notice that u_l satisfies $Lu_l \geq 0$ in Ω. Indeed, for arbitrary $v \in H_0^1(\Omega)$ such that $v \geq 0$ in Ω we have $u_l + v \in K_l$. Thus we can write

$$a(u_l, u_l + v - u_l) \geq 0.$$

The above inequality gives that $a(u_l, v) \geq 0$ for arbitrary $v \in H_0^1(\Omega)$ such that $v \geq 0$ in Ω.

We have $\tilde{\tilde{\Psi}}_i = \Psi_i$ in E_i, $L\tilde{\Psi}_i = 0$ in $\Omega \backslash E_i$, $\tilde{\Psi}_i\big|_{\partial\Omega} = 0$ and $Lu_l \geq 0$ in $\Omega \backslash E_i$. Therefore using the maximum principle we derive that $u_l \geq \tilde{\Psi}_i$ in Ω.

Moreover, we have that $u_l \geq \Psi$ in Ω. It is true due to the fact that $u_l \geq \tilde{\Psi}_i$ in Ω for $i = 1, \ldots, l$. This together with $u_l \in H_0^1(\Omega)$ implies that $u_l \in \tilde{K}_1$ where $\tilde{K}_1$ is defined in (11).

Let us denote by $\tilde{u}_1$ the solution to GP with the impediment Ψ and f being equal to zero. Since $u_l \in \tilde{K}_1$ we can state that

$$a(\tilde{u}_1, u_l - \tilde{u}_1) \geq 0.$$

On the other hand, $\tilde{K}_1 \subset K_l$ since if $v \in \tilde{K}_1$ then $v \in H_0^1(\Omega)$ and $v \geq \Psi \geq \Psi_i$ on E_i for $i = 1, \ldots, l$. Hence we can write

$$a(u_l, \tilde{u}_1 - u_l) \geq 0,$$

because $u_l \in K_l$ solves the variational inequality (8) with $f = 0$. Having added the last two inequalities we shall obtain (using the coercivity of the form $a(\cdot, \cdot)$ in $H_0^1(\Omega)$) that there exists $\nu > 0$ such that

$$\nu \parallel \tilde{u}_1 - u_l \parallel^2 \leq a(\tilde{u}_1 - u_l, \tilde{u}_1 - u_l) \leq 0,$$

which implies that $\tilde{u}_1 = u_l$ in Ω. Lipschitz continuity of $\tilde{u}_1$ (see Lemma 3.1) completes the proof. $\square$

The result similar to the one presented in Theorem 3.2 can be obtained for $m - IIP$ $(m \geq 1)$.

Theorem 3.3. *If $\Phi_i \in H^{1,\infty}(F_i)$ then there exists a unique solution u^m to $m - IIP$ with the impediments Φ_i $(i = 1, \ldots, m)$. Moreover, if $f = 0$ this solution is Lipschitz continuous.*

The proof of Theorem 3.3 is almost identical to the previous one, so we omit it.

In the third theorem we consider $1, 1 - DIP$. In this case we extend the result given in [SV] by adding one impediments from above.

Theorem 3.4. *If the functions $\Psi_1 \in H^{1,\infty}(E_1)$, $\Phi_1 \in H^{1,\infty}(F_1)$ satisfy*

$$\Psi_1 \leq \Phi_1 \quad \text{in } E_1 \cap F_1, \tag{16}$$

then there exists a unique solution u_1^1 *to* $1,1-DIP$ *with the impediments* Ψ_1, Φ_1. *Moreover, if* $f=0$ *this solution is Lipschitz continuous.*

Proof. Firstly, applying the ideas presented in the proof of Theorem 3.2 (see (14)) we construct the Lipschitz extensions of Ψ_1 and Φ_1 onto the whole domain Ω. In order to reduce the complexity of notations those extensions we still call Ψ_1 and Φ_1. Then we observe that $\max\{\Psi_1,0\}+\min\{\Phi_1,0\}\in K_1^1$ which gives us existence of the unique solution $u_1^1\in K_1^1$ to the $1,1-DIP$.

In the proof we shall construct two Lipschitz functions Ψ and Φ such that the solution $\tilde{u}_1^1$ of DGP with the impediments Ψ and Φ and f being equal to zero will coincide with u_1^1 – the solution of $1,1-DIP$.

Let us consider the coincidence set $I[u_1^1]$ for $1,1-DIP$. Obviously it is contained in $E_1\cup F_1$ (as (16) holds). We denote by $I_F[u_1^1]$ that part of $I[u_1^1]$ which is contained in $F_1\cap\overline{(\Omega\backslash E_1)}$ where $u_1^1=\Phi_1$ and by $I_E[u_1^1]$ that part of $I[u_1^1]$ which is contained in $E_1\cap\overline{(\Omega\backslash F_1)}$ where $u_1^1=\Psi_1$. Now we define

$$\tilde{\Psi}_1=\begin{cases}\Psi_1 & \text{in } E_1\\ \min\{\Psi_1,\Phi_1\} & \text{in } \Omega\backslash E_1,\end{cases}\tag{17}$$

$$\tilde{\Phi}_1=\begin{cases}\Phi_1 & \text{in } F_1\\ \max\{\Psi_1,\Phi_1\} & \text{in } \Omega\backslash F_1.\end{cases}\tag{18}$$

Both functions $\tilde{\Psi}_1$ and $\tilde{\Phi}_1$ are continuous. Moreover, they are both Lipschitz.

Now let us take a Lipschitz function $\xi\in H_0^{1,\infty}(\Omega)$ such that $\xi|_{\partial(\Omega\backslash E_1)}=0$ and $\xi<0$ in $\Omega\backslash E_1$. Next we consider the Lipschitz function $\delta\in H^{1,\infty}(\Omega)$ where we put $\delta=\tilde{\Psi}_1+\xi$. We know that

$$u_1^1=\Phi_1=\tilde{\Phi}_1 \quad \text{in } \mathrm{I_F}[\mathrm{u_1^1}].\tag{19}$$

It also satisfies

$$u_1^1>\delta=\tilde{\Psi}_1+\xi \quad \text{in } \mathrm{I_F}[\mathrm{u_1^1}]\tag{20}$$

since $\tilde{\Psi}_1+\xi=\min\{\Psi_1,\Phi_1\}+\xi<\tilde{\Phi}_1$ in $I_F[u_1^1]$. From the continuity of u_1^1, ξ and $\tilde{\Psi}_1$ we state that there exists a neighbourhood O_F of $I_F[u_1^1]$ where the inequality (20) holds.

Now we choose a set D_F with the smooth boundary in the following way:

$$I_F[u_1^1]\subset D_F\subset\bar{D}_F\subset O_F\cap\overline{(\Omega\backslash E_1)}.$$

Let ψ be the solution of the problem:

$$\begin{cases} L\psi = 0 & \text{in } (\Omega\backslash E_1)\backslash\bar{D}_F \\ \psi = \delta & \text{in } \partial((\Omega\backslash E_1)\backslash\bar{D}_F). \end{cases} \tag{21}$$

The function ψ is Lipschitz (see [ADN], [BC]). Next we remark that the set D was chosen in such a way that $(\Omega\backslash E_1)\backslash\bar{D}_F \subset \Omega\backslash I[u_1^1]$. Therefore using again the basic properties of the solutions to the obstacle problems (see [KS]) we have that $Lu_1^1 = 0$ in $(\Omega\backslash E_1)\backslash\bar{D}_F$. Hence

$$L(u_1^1 - \psi) = 0 \quad \text{in } (\Omega\backslash \mathrm{E}_1)\backslash\bar{\mathrm{D}}_\mathrm{F}.$$

Moreover, we have

$$u_1^1 = \psi = \Psi_1 \quad \text{in } \partial\Omega,$$

which follows from the definition of extension of Ψ_1 and the constructions of $\tilde{\Psi}_1$ and ξ,

$$u_1^1 \geq \psi = \Psi_1 \quad \text{in } \partial E_1,$$

which follows from (16) and the constructions of $\tilde{\Psi}_1$ and ξ,

$$u_1^1 \geq \psi \quad \text{in } \partial D_F,$$

which follows from (20) and the construction of the set D_F. Then the maximum principle implies that

$$u_1^1 \geq \psi \quad \text{in } (\Omega\backslash \mathrm{E}_1)\backslash\bar{\mathrm{D}}_\mathrm{F}. \tag{22}$$

Finally we put

$$\Psi = \begin{cases} \Psi_1 & \text{in } E_1 \\ \delta & \text{in } \bar{D}_F \\ \psi & \text{in } (\Omega\backslash E_1)\backslash\bar{D}_F. \end{cases} \tag{23}$$

Clearly the function Ψ is Lipschitz continuous in Ω. Moreover,

$$u_1^1 \geq \Psi \quad \text{in } \Omega \tag{24}$$

since $u_1^1 \in K_1^1$, (22) holds and (20) is satisfied in $D \subset O_F \cap \overline{(\Omega\backslash E_1)}$.

Now we pass to the remaining part of the proof. We choose a Lipschitz function $\eta \in H_0^{1,\infty}(\Omega)$ such that $\eta > 0$ in $\Omega\backslash F_1$ and $\eta|_{\partial(\Omega\backslash F_1)} = 0$. Next we consider the Lipschitz function $\sigma \in H^{1,\infty}(\Omega)$ where we put $\sigma = \tilde{\Phi}_1 + \eta$. We know that $u_1^1 = \tilde{\Psi}_1 = \Psi_1$ in $I_E[u_1^1]$. It also satisfies the following:

$$u_1^1 < \sigma = \tilde{\Phi}_1 + \eta \quad \text{in } I_E[u_1^1] \tag{25}$$

as $\tilde{\Psi}_1 = \Psi < \tilde{\Phi}_1 + \eta$ in $I_E[u_1^1]$. From the continuity of u_1^1, η and $\tilde{\Phi}_1$ we state that there exists a neighbourhood O_E of $I_E[u_1^1]$ where the inequality (25) holds. Acting similarly as above we can choose a set D_E with smooth boundary such that:

$$I_E[u_1^1] \subset D_E \subset \bar{D}_E \subset O_E \cap (\Omega \backslash F_1).$$

Denoting by ϕ the solution of the problem

$$\begin{cases} L\phi = 0 & \text{in } (\Omega\backslash F_1)\backslash \bar{D}_E \\ \phi = \sigma & \text{in } \partial((\Omega\backslash F_1)\backslash \bar{D}_E), \end{cases} \tag{26}$$

we get that ϕ is Lipschitz provided σ is Lipschitz. Knowing that $(\Omega\backslash F_1)\backslash \bar{D}_E \subset \Omega\backslash I[u_1^1]$ and using the maximum principle we deduce that:

$$u_1^1 \leq \phi \quad \text{ in } (\Omega\backslash \mathrm{F}_1)\backslash \bar{\mathrm{D}}_\mathrm{E}. \tag{27}$$

Finally we put

$$\Phi = \begin{cases} \Phi_1 & \text{in } F_1 \\ \sigma & \text{in } \bar{D}_E \\ \phi & \text{in } (\Omega\backslash F_1)\backslash \bar{D}_E. \end{cases} \tag{28}$$

Clearly the function Φ is Lipschitz. Moreover,

$$u_1^1 \leq \Phi \quad \text{ in } \Omega \tag{29}$$

as $u_1^1 \in K_1^1$, (27) holds and (25) is satisfied in $D_E \subset O_E \cap \overline{\Omega\backslash F_1}$.

Conditions (24), (29) together with $u_1^1 \in H_0^1(\Omega)$ imply that $u_1^1 \in \tilde{K}_1^1$, where $\tilde{K}_1^1$ is defined in (12). Moreover, $\tilde{K}_1^1 \subset K_1^1$. Indeed, if $v \in \tilde{K}_1^1$ then $v \in H_0^1(\Omega)$, $v \geq \Psi = \Psi_1$ in E_1 and $v \leq \Phi = \Phi_1$ in F_1 which gives that $v \in K_1^1$.

Let us denote by $\tilde{u}_1^1$ the solution of the DGP with the impediments Ψ, Φ given by (23), (28), respectively and f being equal to zero. Using coercivity of the form $a(\cdot,\cdot)$ on $H_0^1(\Omega)$ we shall deduce that $u_1^1 = \tilde{u}_1^1$. This completes the proof (as $\tilde{u}_1^1$ is Lipschitz — see Lemma 3.1). □

The last theorem describes the Lipschitz continuity of the solutions in the most general case, i.e., $l, m - DIP$.

Theorem 3.5. *If the functions $\Psi_i \in H^{1,\infty}(E_i)$ (i=1,...,l), $\Phi_j \in H^{1,\infty}(F_j)$ (j=1,...,m) satisfy*

$$\Psi_i \leq \Phi_j \quad \text{ in } E_i \cap F_j, \tag{30}$$

then there exists a unique solution u_l^m *to* $l, m - DIP$ *with the impediments* Ψ_i, Φ_j. *Moreover, if* $f = 0$ *this solution is Lipschitz continuous.*

Proof. At the beginning similarly to what we did in the proof of Theorem 3.2 (see (14)) we construct the Lipschitz extensions of Ψ_i and Φ_j onto the whole domain Ω. Again in order to avoid too much complexity these extensions will be still called Ψ_i and Φ_j. Then we observe that $\max\{\Psi_1, \dots, \Psi_l, 0\} + \min\{\Phi_1, \dots, \Phi_m, 0\} \in K_l^m$ which gives us existence of the unique solution $u_l^m \in K_l^m$ to $l, m - DIP$.

It could be proved (see [ES]) that the following estimates for the solutions to the inner obstacle problems are satisfied

$$u_{1(i)}^m \le u_l^m \le u_l^{1(j)} \quad \text{in } \Omega, \tag{31}$$

where $u_{1(i)}^m$ denotes the solution of $1, m - DIP$ with an arbitrarily fixed impediment Ψ_i and m impediments Φ_j, $u_l^{1(j)}$ is the solution of $l, 1 - DIP$ with l impediments Ψ_i and an arbitrarily fixed Φ_j.

Firstly, we examine $1, m - DIP$ with the fixed impediment Ψ_i and m impediments Φ_j. Using the construction presented in the proof of Theorem 3.4 for each pair of the obstacles (Ψ_i, Φ_j) we construct m global obstacles $\tilde{\Phi}_j$ according to (28) and next we put $\Phi^i = \min_{1 \le j \le m} \tilde{\Phi}_j$. Then we consider $1, 1 - DIP$ with the admissible set

$$K_{1(i)}^1 = \{v \in H_0^1(\Omega) \ : \ v \ge \Psi_i \text{ on } E_i \wedge v \le \Phi^i \text{ on } \Omega\}. \tag{32}$$

Its solution exists and we denote it by $u_{1(i)}^1$.

It can be easily seen that $u_{1(i)}^m \in K_{1(i)}^1$ since (29) is satisfied for all $j = 1, \dots, m$. On the other hand, $K_{1(i)}^1 \subset K_{1(i)}^m$. Indeed, if we take $v \in K_{1(i)}^1$ then $v \ge \Psi_i$ on E_i and $v \le \Phi^i = \min_{1 \le j \le m} \tilde{\Phi}_j$ and $\tilde{\Phi}_j = \Phi_j$ on F_j. Using the coerciveness of the form $a(\cdot, \cdot)$ on $H_0^1(\Omega)$ we get that $u_{1(i)}^m = u_{1(i)}^1$.

Having $1, 1 - DIP$ with the pair of obstacles (Ψ_i, Φ^i) we apply once again the construction described in the proof of the previous theorem and we create the function $\bar{\Psi}_i$ according to (23). Obviously

$$u_{1(i)}^m \ge \bar{\Psi}_i \quad \text{on } \Omega. \tag{33}$$

Then we consider $l, 1 - DIP$ with the fixed impediment Φ_j. For each pair of the obstacles (Ψ_i, Φ_j) we build l global obstacles $\tilde{\Psi}_i$ according to (23) and next we put $\Psi^j = \max_{1 \le i \le l} \tilde{\Psi}_i$. Then we consider

$1, 1-DIP$ with the admissible set

$$K_1^{1(j)} = \{v \in H_0^1(\Omega) \ : \ v \leq \Psi^j \text{ on } \Omega \wedge v \leq \Phi_j \text{ on } F_j\}. \tag{34}$$

Its solution exists and we call it $u_1^{1(j)}$.

One can observe that $u_l^{1(j)} \in K_1^{1(j)}$ since (24) is satisfied for all $i = 1, \ldots, l$. Moreover, $K_1^{1(j)} \subset K_l^{1(j)}$. Indeed, if we take $v \in K_1^{1(j)}$ then $v \leq \Phi_j$ on F_j and $v \geq \Psi^j = \max_{1 \leq i \leq l} \tilde{\Psi}_i$ and $\tilde{\Psi}_i = \Psi_i$ on E_i. Using the coerciveness of the form $a(\cdot,\cdot)$ on $H_0^1(\Omega)$ we get that $u_l^{1(j)} = u_1^{1(j)}$.

Concentrating once again on $1, 1-DIP$ with the pair of obstacles (Ψ^j, Φ_j) we create using (28) the global impediment $\bar{\Phi}_j$ such that

$$u_l^{1(j)} \leq \bar{\Phi}_j \quad \text{ on } \Omega. \tag{35}$$

At this moment we deal with two Lipschitz function $\bar{\Psi}_i$, $\bar{\Phi}_j$ satisfying (33), (35), respectively. Repeating for all $i = 1, \ldots, l$ and for all $j = 1, \ldots, m$ the constructions described above we can build l functions $\bar{\Psi}_i$ $(i = 1, \ldots, l)$ and m functions $\bar{\Phi}_j$ $(j = 1, \ldots, m)$. Then we define:

$$\Psi = \max_{i=1,\ldots,l} \bar{\Psi}_i, \qquad \Phi = \min_{j=1,\ldots,m} \bar{\Phi}_j.$$

From (31) we get the following estimates which hold for arbitrary i, j

$$\bar{\Psi}_i \leq u_{1(i)}^m \leq u_l^m \leq u_l^{1(j)} \leq \bar{\Phi}_j.$$

Thus

$$\Psi \leq u_l^m \leq \Phi. \tag{36}$$

This together with $u_l^m \in H_0^1(\Omega)$ implies that $u_l^m \in \tilde{K}_1^1$, where $\tilde{K}_1^1$ is defined in (13). Moreover, due to definition of Ψ and Φ, it is easy to see that $\tilde{K}_1^1 \subset K_l^m$.

Let us denote by $\tilde{u}_1^1$ the solution of DGP with the impediments Ψ and Φ. Using the coercivity of the form $a(\cdot,\cdot)$ in $H_0^1(\Omega)$ we deduce that u_l^m equals $\tilde{u}_1^1$ which is Lipschitz continuous (see Lemma 3.1). This completes the proof. □

Remark 3.6. It is well known that in case of solutions of the global problems one can expect their regularity up to $H^{2,p}$. For the inner problem the situation is much more complicated. Despite $H^{2,p}$ regularity of the obstacle the same class of the solution can not be obtained. However under certain assumptions it is possible to get $H^{2,p}$ regularity of the solutions (see [BS], [JOS]).

Remark 3.7. It is worth pointing out that it was possible to adopt the method of identification of the inner problem with the global one to obtain the results concerning continuous dependence on obstacle of solutions (see [OS], [JOS]).

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