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# Theta Functions

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ABSTRACT. On our analytic way to the group structure of an elliptic function we meet so called theta functions. These complex functions are entire and quasi-periodic with respect to a lattice $\Lambda$. In the proof of Abel's theorem we use their properties to characterise all meromorphic functions $f$ from $\mathbb{C}/\Lambda$ to $\mathbb{C}$. Finally we have a closer look at a very special and interesting $\Lambda$-periodic meromorphic function, the Weierstraß $\wp$-function. This function delivers an analytic way to give a group structure to an algebraic variety.

## 1. Introduction

First of all, we want to analyse periodic complex functions $f : \mathbb{C} \to \mathbb{C}$ with respect to a lattice $\Lambda$. So let us fix once and for all a complex number $\tau \in \mathbb{C}$, Im $\tau > 0$ and consider the lattice $\Lambda := \mathbb{Z} \oplus \tau\mathbb{Z} \subset \mathbb{C}$.

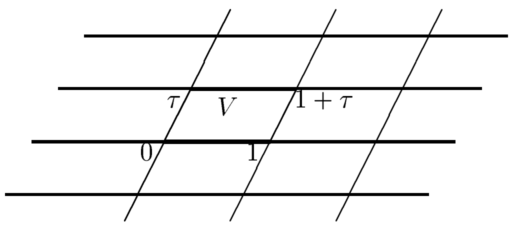


FIGURE 1. The lattice $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$ and its fundamental parallelogram $V = \{z = t_1 + t_2\tau \in \mathbb{C} : 0 \leq t_1, t_2 < 1\}$.

**Lemma 1.** *An entire doubly-periodic complex function is constant.*

To prove this lemma we need Liouville's Theorem, which we know from complex analysis. It states that each entire and bounded complex function $f : \mathbb{C} \to \mathbb{C}$ is constant.

*Proof.* The values of a doubly-periodic function are completely determined by the values on the closure of the fundamental parallelogram $\overline{V} = \{z \in \mathbb{C} : z = t_1 + t_2\tau \text{ for some } 0 \leq t_1, t_2 \leq 1\}$ which is a compact set. But a continuous function on a compact set is bounded. Hence our function is entire and bounded. Therefore it is constant by Liouville's Theorem. □

As we have seen, *entire doubly-periodic* functions are not very interesting, so in the following we will consider *entire quasi-periodic* functions and use them to prove Abel's Theorem which says what *meromorphic doubly-periodic* functions look like.

## 2. Theta Functions and Abel's Theorem

**Definition.** The *basic theta function* is defined to be the function $\theta : \mathbb{C} \to \mathbb{C}$ given by

$$\theta(z) := \theta(\tau)(z) := \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 \tau) \exp(2\pi i n z)$$

**Note.** The function $\theta$ depends on $\tau$. So for each $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$ we get a (not necessarily different) basic theta function. Hence there is a whole family of basic theta functions $\{\theta(\tau)\}_{\tau\in\mathbb{C},\operatorname{Im}\,\tau>0}$. But here we assume $\tau$ to be fixed, so we have only one basic theta function.

**Remark.** As the series in the definition above is locally uniformly unordered convergent (without proof) our basic theta function is an entire function.

**Lemma 2.** *The basic theta function is quasi-periodic.*

*Proof.* Consider $\theta(z + \lambda)$ for $\lambda \in \Lambda$, i.e. $\lambda = p\tau + q$ for $p, q \in \mathbb{Z}$.
For $\lambda = 1$, i.e. for $p = 0$ and $q = 1$ we have

$$\begin{aligned}\theta(z+1) &\overset{def}{=} \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau)\exp(2\pi i n(z+1)) \\ &= \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau + 2\pi i n z + 2\pi i n) \\ &= \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau)\exp(2\pi i n z) \underbrace{\exp(2\pi i n)}_{=1 \text{ for all } n\in\mathbb{Z}}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 \tau) \exp(2\pi i n z) \\
&\stackrel{def}{=} \theta(z)
\end{aligned}
$$

Hence the basic theta function is periodic with respect to the $x$-direction.

For $\lambda = \tau$, i.e., for $p = 1$ and $q = 0$ we have

$$
\begin{aligned}
\theta(z+\tau) &\stackrel{def}{=} \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 \tau) \exp(2\pi i n(z+\tau)) \\
&= \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n z + 2\pi i n \tau)
\end{aligned}
$$

if we complete the square and rearrange the summands then

$$
\begin{aligned}
&= \sum_{n\in\mathbb{Z}} \exp\left(\pi i n^2 \tau + 2\pi i n \tau + \pi i \tau - \pi i \tau \right. \\
&\qquad \left. + 2\pi i n z + 2\pi i z - 2\pi i z\right) \\
&= \exp(-\pi i \tau - 2\pi i z) \sum_{n\in\mathbb{Z}} \exp(\pi i (n+1)^2 \tau) \exp(2\pi i (n+1) z)
\end{aligned}
$$

if we make a simple index shift $m = n + 1$ then

$$
\begin{aligned}
&= \exp(-\pi i \tau - 2\pi i z) \sum_{m\in\mathbb{Z}} \exp(\pi i m^2 \tau) \exp(2\pi i m z) \\
&\stackrel{def}{=} \exp(-\pi i \tau - 2\pi i z)\theta(z)
\end{aligned}
$$

Hence the basic theta function is not periodic with respect to the $\tau$-direction as in general $\exp(-\pi i \tau - 2\pi i z) \neq 1$.

In the general case we obtain

$$
\begin{aligned}
\theta(z+\lambda) &= \theta(z + p\tau + q) \\
&\stackrel{def}{=} \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 \tau) \exp(2\pi i n(z + p\tau + q)) \\
&= \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n z + 2\pi i n p \tau + 2\pi i n q)
\end{aligned}
$$

if we complete the square and rearrange the summands then

$$\begin{aligned}
&= \sum_{n\in\mathbb{Z}} \exp\big(\pi i n^2\tau + 2\pi i n p\tau + \pi i p^2\tau - \pi i p^2\tau \\
&\qquad +2\pi i n z + 2\pi i p z - 2\pi i p z + 2\pi i n q\big) \\
&= \exp(-\pi i p^2\tau - 2\pi i p z) \\
&\qquad \cdot \sum_{n\in\mathbb{Z}} \Big[ \exp(\pi i (n+p)^2\tau)\exp(2\pi i(n+p)z) \\
&\qquad\qquad \underbrace{\exp(2\pi i n q)}_{=1 \text{ for all } n\in\mathbb{Z}} \Big] \\
&= \exp(-\pi i p^2\tau - 2\pi i p z) \\
&\qquad \cdot \sum_{n\in\mathbb{Z}} \exp(\pi i (n+p)^2\tau)\exp(2\pi i(n+p)z)
\end{aligned}$$

if we make a simple index shift $m = n + p$ then

$$\begin{aligned}
&= \exp(-\pi i p^2\tau - 2\pi i p z) \sum_{m\in\mathbb{Z}} \exp(\pi i m^2\tau)\exp(2\pi i m z) \\
&\stackrel{def}{=} \exp(-\pi i p^2\tau - 2\pi i p z)\theta(z)
\end{aligned}$$

Hence the basic theta function $\theta$ is quasi-periodic with

$$\begin{aligned}
\theta(z+\lambda) &= \theta(z + p\tau + q) \\
&= \exp(-\pi i p^2\tau - 2\pi i p z)\theta(z)
\end{aligned}$$

for all $\lambda = p\tau + q \in \Lambda$ and $z \in \mathbb{C}$. □

**Definition.** We define

$$e(\lambda, z) := \exp(-\pi i p^2\tau - 2\pi i p z)$$

and call this the *automorphy factor*.

**Remark.** We have $e(\lambda_1 + \lambda_2, z) = e(\lambda_1, z + \lambda_2)e(\lambda_2, z)$ for all $\lambda_1$, $\lambda_2 \in \Lambda$.

Let $\lambda_1$, $\lambda_2 \in \Lambda$, i.e. $\lambda_1 = p_1\tau + q_1$ and $\lambda_2 = p_2\tau + q_2$ for some $p_1$, $p_2$, $q_1$, $q_2 \in \mathbb{Z}$, and thus $\lambda_1 + \lambda_2 = (p_1 + p_2)\tau + (q_1 + q_2) \in \Lambda$. Then

$$
\begin{aligned}
e(\lambda_1 + \lambda_2, z) &= e((p_1 + p_2)\tau + (q_1 + q_2), z) \\
&\stackrel{def}{=} \exp(-\pi i(p_1 + p_2)^2\tau - 2\pi i(p_1 + p_2)z)) \\
&= \exp(-\pi i p_1^2\tau - 2\pi i p_1 p_2\tau - \pi i p_2^2\tau - 2\pi i p_1 z - 2\pi i p_2 z) \\
&\stackrel{def}{=} \exp(-\pi i p_1^2\tau - 2\pi i p_1 p_2\tau - 2\pi i p_1 z)e(\lambda_2, z) \\
&= \exp(-\pi i p_1^2\tau - 2\pi i p_1 z - 2\pi i p_1 p_2\tau - \underbrace{2\pi i p_1 q_2}_{\exp(2\pi i p_1 q_2)=1}) \\
&\qquad \cdot e(\lambda_2, z) \\
&= \exp(-\pi i p_1^2\tau - 2\pi i p_1(z + \lambda_2))e(\lambda_2, z) \\
&\stackrel{def}{=} e(\lambda_1, z + \lambda_2)e(\lambda_2, z)
\end{aligned}
$$

**Summary.** The basic theta function $\theta : \mathbb{C} \to \mathbb{C}$ is entire and quasi-periodic with automorphy factor $e$, i.e., we have

$$\theta(z + \lambda) = e(\lambda, z)\theta(z) = \exp(-\pi i p^2\tau - 2\pi i p z)\theta(z) \tag{1}$$

for all $\lambda = p\tau + q \in \Lambda$ and all $z \in \mathbb{C}$.

Now we want to enlarge our category of theta functions. So far we have only one (basic) theta function corresponding to the point $0 \in \mathbb{C}$ (and each point $q \in \mathbb{Z} \subset \mathbb{C}$). Now, for our fixed $\tau$, we will define a new theta function for each point in $\mathbb{C}$. Therefore let's start with our old theta function and translate z by a fixed $\xi$, i.e. consider $\theta(z + \xi)$ for $\xi = a\tau + b$ for some fixed $a$, $b \in \mathbb{R}$:

$$
\begin{aligned}
\theta(z + \xi) &= \theta(z + a\tau + b) \\
&\stackrel{def}{=} \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau)\exp(2\pi i n(z + a\tau + b)) \\
&= \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau + 2\pi i n z + 2\pi i n a\tau + 2\pi i n b)
\end{aligned}
$$

If we complete the square and rearrange the summands then we obtain

$$\begin{aligned}\theta(z+\xi) = \sum_{n\in\mathbb{Z}} &\exp\big(\pi i n^2\tau + 2\pi i n a\tau + \pi i a^2\tau - \pi i a^2\tau \\ &\quad +2\pi i n(z+b) + 2\pi i a(z+b) - 2\pi i a(z+b)\big) \\ = {} &\exp(-\pi i a^2\tau - 2\pi i a(z+b)) \\ &\quad \cdot \sum_{n\in\mathbb{Z}} \exp(\pi i(n+a)^2\tau)\exp(2\pi i(n+a)(z+b))\end{aligned}$$

Note that the sum $\sum_{n\in\mathbb{Z}} \exp(\pi i(n+a)^2\tau)\exp(2\pi i(n+a)(z+b))$ looks very similar to the sum in the definition of our basic theta function above.

**Definition.** For $\xi = a\tau + b$ and $a, b \in \mathbb{R}$ the *modified theta function* is defined to be the function $\theta_\xi : \mathbb{C} \to \mathbb{C}$ given by

$$\theta_\xi(z) := \theta_\xi(\tau)(z) := \sum_{n\in\mathbb{Z}} \exp(\pi i(n+a)^2\tau)\exp(2\pi i(n+a)(z+b))$$

and $\xi$ is called *theta characteristic.*

**Note.** From the calculation above we obtain a relation between the basic theta function and the modified theta function with characteristic $\xi = a\tau + b$ for some fixed $a, b \in \mathbb{R}$:

$$\begin{aligned}\theta_\xi(z) &= \sum_{n\in\mathbb{Z}} \exp(\pi i(n+a)^2\tau)\exp(2\pi i(n+a)(z+b)) && (2)\\ &= \exp(\pi i a^2\tau + 2\pi i a(z+b))\theta(z+\xi) && (3)\end{aligned}$$

for all $z \in \mathbb{C}$.

**Remark.** As the series in the definition is locally uniformly unordered convergent (without proof) the modified theta functions are entire functions.

**Lemma 3.** *Modified theta functions are quasi-periodic functions.*

*Proof.* Let $a$, $b \in \mathbb{R}$ such that $\xi = a\tau + b$ is the characteristic of the modified theta function $\theta_\xi$. Consider $\theta_\xi(z+\lambda)$ for $\lambda = p\tau + q \in \Lambda$.

$$\begin{aligned}
\theta_\xi(z+\lambda) &\overset{(3)}{=} \exp(\pi i a^2\tau + 2\pi i a(z+\lambda+b))\theta(z+\lambda+\xi) \\
&\overset{(1)}{=} \exp(\pi i a^2\tau + 2\pi i a(z+\lambda+b))e(\lambda, z+\xi)\theta(z+\xi) \\
&\overset{(3)}{=} \exp(\pi i a^2\tau + 2\pi i a(z+\lambda+b))e(\lambda, z+\xi) \\
&\qquad \cdot \exp(-\pi i a^2\tau - 2\pi i a(z+b))\theta_\xi(z) \\
&= \exp(2\pi i a\lambda)\exp(-\pi i p^2\tau - 2\pi i p(z+\xi))\theta_\xi(z) \\
&= \exp(2\pi i a\lambda - \pi i p^2\tau - 2\pi i p(z+\xi))\theta_\xi(z)
\end{aligned}$$

Hence the modified theta function $\theta_\xi$ is quasi-periodic with

$$\begin{aligned}
\theta_\xi(z+\lambda) &= \theta_{a\tau+b}(z+p\tau+q) \\
&= \exp(2\pi i a\lambda - \pi i p^2\tau - 2\pi i p(z+\xi))\theta_\xi(z)
\end{aligned}$$

for all $\lambda = p\tau + q \in \Lambda$ and $z \in \mathbb{C}$. □

**Definition.** Let $a$, $b \in \mathbb{R}$ be fixed and let $\xi = a\tau + b$. We define

$$e_\xi(\lambda, z) := \exp(2\pi i a\lambda - \pi i p^2\tau - 2\pi i p(z+\xi))$$

and call this the *automorphy factor*.

**Remark.** Let $a$, $b \in \mathbb{R}$ be fixed and let $\xi = a\tau + b$. We have $e_\xi(\lambda_1+\lambda_2, z) = e_\xi(\lambda_1, z+\lambda_2)e_\xi(\lambda_2, z)$ for all $\lambda_1$, $\lambda_2 \in \Lambda$.

Let $\lambda_1$, $\lambda_2 \in \Lambda$, i.e. $\lambda_1 = p_1\tau + q_1$ and $\lambda_2 = p_2\tau + q_2$ for some $p_1$, $p_2$, $q_1$, $q_2 \in \mathbb{Z}$, and $\lambda_1 + \lambda_2 = (p_1+p_2)\tau + (q_1+q_2) \in \Lambda$. Then

$$\begin{aligned}
e_\xi(\lambda_1+\lambda_2, z) &= e_\xi((p_1+p_2)\tau + (q_1+q_2), z) \\
&\overset{def}{=} \exp\big(2\pi i a(\lambda_1+\lambda_2) - \pi i(p_1+p_2)^2\tau \\
&\qquad -2\pi i(p_1+p_2)(z+\xi)\big) \\
&= \exp\big(2\pi i a\lambda_1 + 2\pi i a\lambda_2 - \pi i p_1^2\tau - 2\pi i p_1p_2\tau - \pi i p_2^2\tau \\
&\qquad -2\pi i p_1(z+\xi) - 2\pi i p_2(z+\xi)\big)
\end{aligned}$$

$$
\begin{aligned}
&\overset{def}{=} \exp(2\pi i a\lambda_1 - \pi i p_1^2\tau - 2\pi i p_1 p_2 \tau - 2\pi i p_1(z+\xi)) \\
&\qquad e_\xi(\lambda_2, z) \\
&= \exp\big(2\pi i a\lambda_1 - \pi i p_1^2\tau - 2\pi i p_1 p_2\tau - \underbrace{2\pi i p_1 q_2}_{\exp(2\pi i p_1 q_2)=1} \\
&\qquad -2\pi i p_1(z+\xi)\big)\, e_\xi(\lambda_2, z) \\
&= \exp(2\pi i a\lambda_1 - \pi i p_1^2\tau - 2\pi i p_1(z+\lambda_2+\xi)) e_\xi(\lambda_2, z) \\
&\overset{def}{=} e_\xi(\lambda_1, z+\lambda_2) e_\xi(\lambda_2, z)
\end{aligned}
$$

**Summary.** Let $\xi = a\tau + b$ with $a,\ b \in \mathbb{R}$ fixed. The modified theta function with characteristic $\xi$ is entire and quasi-periodic with automorphy factor $e_\xi$, i.e. we have

$$
\begin{aligned}
\theta_\xi(z+\lambda) &= e_\xi(\lambda, z)\theta_\xi(z) && (4) \\
&= \exp(2\pi i a\lambda - \pi i p^2\tau - 2\pi i p(z+\xi))\theta_\xi(z) && (5)
\end{aligned}
$$

for all $\lambda = p\tau + q \in \Lambda$ and all $z \in \mathbb{C}$.

Now we want to determine all zeros of all theta functions. Therefore we consider a special modified theta function, the theta function with characteristic $\sigma := \frac{1}{2}\tau + \frac{1}{2}$. In this case the determination of zeros is very simple because the zeros are easy to describe.

**Lemma 4.** *$\theta_\sigma$ is an odd function, i.e. $\theta_\sigma(-z) = -\theta_\sigma(z)$ for all $z \in \mathbb{C}$. In particular we have $\theta_\sigma(0) = 0$.*

*Proof.* We have

$$
\begin{aligned}
\theta_\sigma(-z) &= \theta_{\frac{1}{2}\tau+\frac{1}{2}}(-z) \\
&\overset{def}{=} \sum_{n\in\mathbb{Z}} \Bigg[\exp\left(\pi i \left(n+\frac{1}{2}\right)^2 \tau\right) \\
&\qquad \exp\left(2\pi i\left(n+\frac{1}{2}\right)\left(-z+\frac{1}{2}\right)\right)\Bigg]
\end{aligned}
$$

if we make a simple index shift $m = -n - 1$ then

$$= \sum_{m \in \mathbb{Z}} \left[ \exp\left(\pi i \left(-m - \frac{1}{2}\right)^2 \tau\right) \right.$$
$$\left. \exp\left(2\pi i \left(-m - \frac{1}{2}\right)\left(-z + \frac{1}{2}\right)\right)\right]$$
$$= \sum_{m \in \mathbb{Z}} \left[ \exp\left(\pi i \left(m + \frac{1}{2}\right)^2 \tau\right) \right.$$
$$\left. \exp\left(2\pi i \left(m + \frac{1}{2}\right)\left(z + \frac{1}{2}\right) - 2\pi i \left(m + \frac{1}{2}\right)\right)\right]$$
$$= \sum_{m \in \mathbb{Z}} \left[ \exp\left(\pi i \left(m + \frac{1}{2}\right)^2 \tau\right) \right.$$
$$\exp\left(2\pi i \left(m + \frac{1}{2}\right)\left(z + \frac{1}{2}\right)\right) \stackrel{def}{=} -\theta_\sigma(z).$$

□

From complex analysis we know a simple way to count zeros and poles of a meromorphic function $f : \mathbb{C} \to \mathbb{C}$:

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f}(z)\, dz = \text{total number of zeros - total number of poles}$$

where $\gamma$ is a piecewise smooth path that runs around each zero and each pole exactly one time. We will use this integral to determine all zeros of the theta functions $\theta_\sigma$ with $\sigma = \frac{1}{2}\tau + \frac{1}{2}$.

**Lemma 5.** *We have $\theta_\sigma(z) = 0$ precisely for all $z \in \Lambda$ and all zeros are simple zeros.*

*Proof.* Consider the fundamental parallelogram $V := \{z \in \mathbb{C} : z = t_1\tau + t_2 \text{ for some } 0 \leq t_1, t_2 < 1\}$. Choose $w \in \mathbb{C}$ such that the border of $V_w := w + V$ contains no zeros of $\theta_\sigma$ and $0 \in V_w$.

Further consider the following paths along the border of $V_w$:

$$\alpha : [0,1] \to \mathbb{C}; t \mapsto w + t$$
$$\beta : [0,1] \to \mathbb{C}; t \mapsto w + 1 + t\tau$$
$$\gamma : [0,1] \to \mathbb{C}; t \mapsto w + (1-t) + \tau$$
$$\delta : [0,1] \to \mathbb{C}; t \mapsto w + (1-t)\tau$$

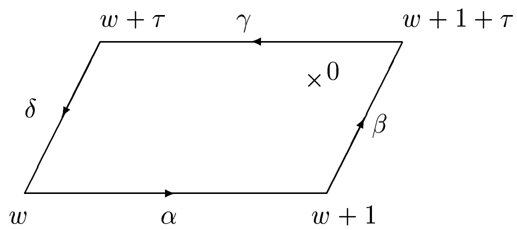


FIGURE 2

In the above figure $w \in \mathbb{C}$ is chosen such that the border of the parallelogram $V_w = w + V$ contains no zeros of $f$ and such that $0 \in V_w$. The paths $\alpha$, $\beta$, $\gamma$ and $\delta$ run along the border of $V_w$. Note that

$$\gamma(t) = w + (1-t) + \tau = \alpha(1-t) + \tau$$

and

$$\delta(t) = w + (1-t)\tau = \beta(1-t) - 1.$$

We want to show that $\frac{1}{2\pi i}\int_{\partial V_w} \frac{\theta'_\sigma}{\theta_\sigma}(z)\, dz = 1.$

Therefore we will show that

$$\frac{1}{2\pi i}\int_\gamma \frac{\theta'_\sigma}{\theta_\sigma}(z)\, dz = 1 - \frac{1}{2\pi i}\int_\alpha \frac{\theta'_\sigma}{\theta_\sigma}(z)\, dz$$

and

$$\frac{1}{2\pi i}\int_\delta \frac{\theta'_\sigma}{\theta_\sigma}(z)\, dz = -\frac{1}{2\pi i}\int_\beta \frac{\theta'_\sigma}{\theta_\sigma}(z)\, dz.$$

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma \frac{\theta'_\sigma}{\theta_\sigma}(z)\, dz &= \frac{1}{2\pi i}\int_0^1 \frac{\theta'_\sigma}{\theta_\sigma}(\gamma(t))\gamma'(t)\, dt \\
&= \frac{1}{2\pi i}\int_0^1 \frac{\theta'_\sigma}{\theta_\sigma}(\alpha(1-t)+\tau)(-1)\, dt \\
&= -\frac{1}{2\pi i}\int_\alpha \frac{\theta'_\sigma}{\theta_\sigma}(z+\tau)\, dz \\
&= -\frac{1}{2\pi i}\int_\alpha \frac{e'_\sigma(\tau,z)\theta_\sigma(z) + e_\sigma(\tau,z)\theta'_\sigma(z)}{e_\sigma(\tau,z)\theta_\sigma(z)}\, dz \\
&= -\frac{1}{2\pi i}\int_\alpha \frac{e'_\sigma(\tau,z)}{e_\sigma(\tau,z)}\, dz - \frac{1}{2\pi i}\int_\alpha \frac{\theta'_\sigma}{\theta_\sigma}(z)\, dz
\end{aligned}$$

when we use $e_\sigma(\tau, z) = \exp(2\pi i \frac{1}{2}\tau - \pi i \tau - 2\pi i(z+\sigma))$ then the above expression becomes

$$\begin{aligned}
-\frac{1}{2\pi i}\int_\alpha \frac{\exp'(-2\pi i(z+\sigma))}{\exp(-2\pi i(z+\sigma))}\,dz - \frac{1}{2\pi i}\int_\alpha \frac{\theta'_\sigma}{\theta_\sigma}(z)\,dz \\
= -\frac{1}{2\pi i}\int_\alpha -2\pi i\,dz - \frac{1}{2\pi i}\int_\alpha \frac{\theta'_\sigma}{\theta_\sigma}(z)\,dz \\
= 1 - \frac{1}{2\pi i}\int_\alpha \frac{\theta'_\sigma}{\theta_\sigma}(z)\,dz
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2\pi i}\int_\delta \frac{\theta'_\sigma}{\theta_\sigma}(z)\,dz &= \frac{1}{2\pi i}\int_0^1 \frac{\theta'_\sigma}{\theta_\sigma}(\delta(t))\delta'(t)\,dt \\
&= \frac{1}{2\pi i}\int_0^1 \frac{\theta'_\sigma}{\theta_\sigma}(\beta(1-t)-1)(-\tau)\,dt \\
&= -\frac{1}{2\pi i}\int_\beta \frac{\theta'_\sigma}{\theta_\sigma}(z-1)\,dz \\
&= -\frac{1}{2\pi i}\int_\beta \frac{e'_\sigma(-1,z)\theta_\sigma(z) + e_\sigma(-1,z)\theta'_\sigma(z)}{e_\sigma(-1,z)\theta_\sigma(z)}\,dz \\
&= -\frac{1}{2\pi i}\int_\beta \frac{e'_\sigma(-1,z)}{e_\sigma(-1,z)}\,dz - \frac{1}{2\pi i}\int_\beta \frac{\theta'_\sigma(z)}{\theta_\sigma(z)}\,dz
\end{aligned}$$

when we use

$$e_\sigma(-1,z) = \exp(-2\pi i\frac{1}{2})$$

then

$$\begin{aligned}
&= -\frac{1}{2\pi i}\int_\beta \frac{\exp'(-\pi i)}{\exp(-\pi i)}\,dz - \frac{1}{2\pi i}\int_\beta \frac{\theta'_\sigma(z)}{\theta_\sigma(z)}\,dz \\
&= -\frac{1}{2\pi i}\int_\beta \frac{\theta'_\sigma(z)}{\theta_\sigma(z)}\,dz
\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial V_w} \frac{\theta'_\sigma}{\theta_\sigma}(z)\,dz &= \frac{1}{2\pi i}\int_\alpha \frac{\theta'_\sigma}{\theta_\sigma}(z)\,dz + \frac{1}{2\pi i}\int_\beta \frac{\theta'_\sigma}{\theta_\sigma}(z)\,dz \\
&\quad + \frac{1}{2\pi i}\int_\gamma \frac{\theta'_\sigma}{\theta_\sigma}(z)\,dz + \frac{1}{2\pi i}\int_\delta \frac{\theta'_\sigma}{\theta_\sigma}(z)\,dz \\
&= 1.
\end{aligned}$$

As $\theta_\sigma$ is holomorphic in $\overline{V_w}$, i.e. it doesn't have any poles, we know that $\theta_\sigma$ has a single zero. And by Lemma 4 this zero is in 0. Now consider $\overline{V}_w + \lambda = \overline{V}_{w+\lambda}$ for some $\lambda \in \Lambda$. As $\theta_\sigma(z+\lambda) = e_\sigma(\lambda, z)\theta_\sigma(z)$ we obtain that $\theta_\sigma$ has the only zero $0 + \lambda = \lambda$ in $\overline{V}_{w+\lambda}$ and this is a simple zero. But $\mathbb{C} = \cup_{\lambda\in\Lambda}\overline{V}_{w+\lambda}$. Hence $\theta_\sigma$ has zeros exactly in $\Lambda$ and all zeros are simple. $\square$

**Corollary 6.** *Let $\xi = a\tau + b$ with $a$, $b \in \mathbb{R}$. We have $\theta_\xi(z) = 0$ precisely for all $z \in \sigma - \xi + \Lambda$ and all its zeros are simple.*

*Proof.* We know $\theta_\sigma(z) = 0$ if and only if $z \in \Lambda$ and all the zeros are simple. Hence

$$\begin{aligned}
\theta_\xi(z) = 0 &\overset{(3)}{\Leftrightarrow} \exp(\pi i a^2\tau + 2\pi i a(z+b))\theta(z+\xi) = 0\\
&\overset{(3)}{\Leftrightarrow} \exp\left(\pi i a^2\tau + 2\pi i a(z+b)\right)\\
&\quad \cdot \exp\Big(-\pi i\Big(\frac{1}{2}\Big)^2\tau - 2\pi i\frac{1}{2}\Big(z + \xi - \frac{1}{2}\tau - \frac{1}{2} + \frac{1}{2}\Big)\Big)\\
&\qquad \cdot \theta_\sigma(z + \xi - \sigma) = 0\\
&\Leftrightarrow z + \xi - \sigma \in \Lambda\\
&\Leftrightarrow z \in \sigma - \xi + \Lambda
\end{aligned}$$

In particular we have $\theta(z) = 0$ if and only if $z \in \sigma + \Lambda$. $\square$

So far we have considered entire quasi-periodic functions. Now we want to use our knowledge about them to see what meromorphic doubly-periodic functions with given zeros $a_i$ and poles $b_j$ of given order $n_i$ resp. $m_j$ and number $n$ resp. $m$ look like. Furthermore we will decide whether such a function exists or not and whether it is unique or not.

**Abel's Theorem 7.** *There is a meromorphic function on $\mathbb{C}/\Lambda$ with zeros $[a_i]$ of order $n_i$ for $1 \le i \le n$ and poles $[b_j]$ of order $m_j$ for $1 \le j \le m$ if and only if $\sum_{i=1}^n n_i = \sum_{j=1}^m m_j$ and $\sum_{i=1}^n n_i[a_i] = \sum_{j=1}^m m_j[b_j]$.*

*Moreover, such a function is unique up to a constant factor.*

*Proof.* "$\Rightarrow$" Let $f : \mathbb{C}/\Lambda \to \mathbb{C}$ be a meromorphic function with zeros $[a_i]$ of order $n_i$ and poles $[b_j]$ of order $m_j$. Choose $w \in \mathbb{C}$ such that $V_w = \{w + z \in \mathbb{C} : z = t_1\tau + t_2 \text{ for some } 0 \le t_1, t_2 < 1\}$ contains a representative $a_i$ resp. $b_j$ for every zero resp. pole of $f$. Further consider the paths

$$\begin{aligned}
\alpha &: [0,1] \to \mathbb{C}; t \mapsto w + t \\
\beta &: [0,1] \to \mathbb{C}; t \mapsto w + 1 + t\tau \\
\gamma &: [0,1] \to \mathbb{C}; t \mapsto w + (1-t) + \tau \\
\delta &: [0,1] \to \mathbb{C}; t \mapsto w + (1-t)\tau
\end{aligned}$$

along the border of $V_w$ and the paths

$$\begin{aligned}
\alpha_i &: [0,1] \to \mathbb{C}; t \mapsto a_i + r_i e^{2\pi i t} \\
\beta_j &: [0,1] \to \mathbb{C}; t \mapsto b_j + s_j e^{2\pi i t}
\end{aligned}$$

around the zeros resp. poles of $f$ where $r_i$ resp. $s_j$ is chosen small enough that $D_i = \{z \in \mathbb{C} : |z - a_i| < r_i\}$ resp. $D'_j = \{z \in \mathbb{C} : |z - b_j| < s_j\}$ contains no other zeros or poles of $f$.

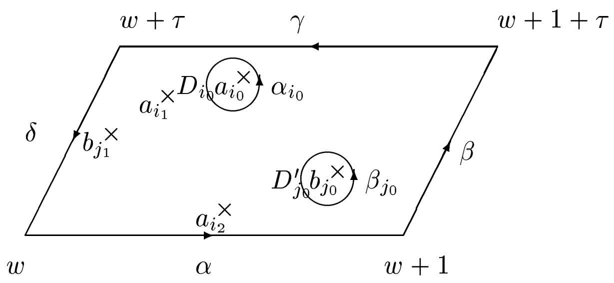


Figure 3

Here, $w \in \mathbb{C}$ is chosen such that the parallelogram $V_w = w + V$ contains a representative $a_i$ resp. $b_j$ for every zero resp. pole of $f$. The paths $\alpha$, $\beta$, $\gamma$ and $\delta$ run along the border of $V_w$, the paths $\alpha_{i_0}$ around the zero $a_{i_0}$ of $f$ and the path $\beta_{j_0}$ around the pole $b_{j_0}$ of $f$.

First we show that $\sum_{i=1}^n n_i a_i - \sum_{j=1}^m m_j b_j \in \Lambda$ as follows:

$$\begin{aligned}
\sum_{i=1}^n n_i a_i - \sum_{j=1}^m m_j b_j &= \sum_{i=1}^n \frac{1}{2\pi i} \int_{\alpha_i} z \frac{f'}{f}(z)\, dz + \sum_{j=1}^m \frac{1}{2\pi i} \int_{\beta_j} z \frac{f'}{f}(z)\, dz \\
&= \frac{1}{2\pi i} \int_{\partial V_w} z \frac{f'}{f}(z)\, dz \in \Lambda
\end{aligned}$$

To establish the first equality note that we can write

$$f(z) = c_i(z - a_i)^{n_i} h_i(z)$$

for a constant $c_i$ and with $h_i(a_i) = 1$ around $a_i$ and hence

$$f'(z) = c_i n_i (z - a_i)^{n_i - 1} \overline{h}_i(z)$$

with $\overline{h}_i(a_i) = 1$. We obtain

$$z\frac{f'}{f}(z) = z\frac{n_i}{z - a_i}\frac{\overline{h}_i}{h_i}(z)$$

with $\frac{\overline{h}_i}{h_i}(a_i) = 1$. Hence we have

$$\frac{1}{2\pi i}\int_{\alpha_i} z\frac{f'}{f}(z)\, dz = n_i a_i$$

by Cauchy's integral formula for discs. The same holds for the poles of $f$.

The second equality is clear since $V_w$ contains a representative for every zero and pole of $f$ in $\mathbb{C}/\Lambda$.

To see, that $\displaystyle\frac{1}{2\pi i}\int_{\partial V_w} z\frac{f'}{f}(z)\, dz$ is an element of $\Lambda$, note that

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\gamma} z\frac{f'}{f}(z)\, dz &= \frac{1}{2\pi i}\int_0^1 \gamma(t)\frac{f'}{f}(\gamma(t))\gamma'(t)\, dt \\
&= \frac{1}{2\pi i}\int_0^1 (\alpha(1-t)+\tau)\frac{f'}{f}((\alpha(1-t)+\tau))(-1)\, dt \\
&= -\frac{1}{2\pi i}\int_0^1 \alpha(1-t)\frac{f'}{f}(\alpha(1-t))\, dt \\
&\qquad - \frac{1}{2\pi i}\int_0^1 \tau\frac{f'}{f}(\alpha(1-t))\, dt \\
&= -\frac{1}{2\pi i}\int_{\alpha} z\frac{f'}{f}(z)\, dz - \tau\frac{1}{2\pi i}\int_{\alpha}\frac{f'}{f}(z)\, dz
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2\pi i}\int_\delta z\frac{f'}{f}(z)\,dz &= \frac{1}{2\pi i}\int_0^1 \delta(t)\frac{f'}{f}(\delta(t))\delta'(t)\,dt \\
&= \frac{1}{2\pi i}\int_0^1 (\beta(1-t)-1)\frac{f'}{f}((\beta(1-t)-1))(-\tau)\,dt \\
&= -\frac{1}{2\pi i}\int_0^1 \beta(1-t)\frac{f'}{f}(\beta(1-t))\tau\,dt \\
&\qquad + \frac{1}{2\pi i}\int_0^1 \frac{f'}{f}(\beta(1-t))\tau\,dt \\
&= -\frac{1}{2\pi i}\int_\beta z\frac{f'}{f}(z)\,dz + \frac{1}{2\pi i}\int_\beta \frac{f'}{f}(z)\,dz
\end{aligned}$$

hence

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial V_w} z\frac{f'}{f}(z)\,dz &= \frac{1}{2\pi i}\int_\alpha z\frac{f'}{f}(z)\,dz + \frac{1}{2\pi i}\int_\beta z\frac{f'}{f}(z)\,dz \\
&\qquad + \frac{1}{2\pi i}\int_\gamma z\frac{f'}{f}(z)\,dz + \frac{1}{2\pi i}\int_\delta z\frac{f'}{f}(z)\,dz \\
&= -\tau\frac{1}{2\pi i}\int_\alpha \frac{f'}{f}(z)\,dz + \frac{1}{2\pi i}\int_\beta \frac{f'}{f}(z)\,dz \in \Lambda
\end{aligned}$$

since $\frac{1}{2\pi i}\int_\beta \frac{f'}{f}(z)\,dz$, $\frac{1}{2\pi i}\int_\alpha \frac{f'}{f}(z)\,dz \in \mathbb{Z}$.

Secondly we show that

$$\begin{aligned}
\sum_{i=1}^n n_i - \sum_{j=1}^m m_j &= \frac{1}{2\pi i}\int_{\partial V_w} \frac{f'}{f}(z)\,dz \\
&= 0
\end{aligned}$$

Again the first equality is clear, since $V_w$ contains a representative for every zero and pole of $f$ in $\mathbb{C}/\Lambda$.

The second equality follows from:

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}(z)\,dz &= \frac{1}{2\pi i}\int_0^1 \frac{f'}{f}(\gamma(t))\gamma'(t)\,dt\\ &= \frac{1}{2\pi i}\int_0^1 \frac{f'}{f}(\alpha(1-t)+\tau)(-1)\,dt\\ &= -\frac{1}{2\pi i}\int_0^1 \frac{f'}{f}(\alpha(1-t))\,dt\\ &= -\frac{1}{2\pi i}\int_\alpha \frac{f'}{f}(z)\,dz\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2\pi i}\int_\delta \frac{f'}{f}(z)\,dz &= \frac{1}{2\pi i}\int_0^1 \frac{f'}{f}(\delta(t))\delta'(t)\,dt\\ &= \frac{1}{2\pi i}\int_0^1 \frac{f'}{f}(\beta(1-t)-1)(-\tau)\,dt\\ &= -\frac{1}{2\pi i}\int_0^1 \frac{f'}{f}(\beta(1-t))\tau\,dt\\ &= -\frac{1}{2\pi i}\int_\beta \frac{f'}{f}(z)\,dz\end{aligned}$$

hence

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial V_w} \frac{f'}{f}(z)\,dz &= \frac{1}{2\pi i}\int_\alpha \frac{f'}{f}(z)\,dz + \frac{1}{2\pi i}\int_\beta \frac{f'}{f}(z)\,dz\\ &\qquad + \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}(z)\,dz + \frac{1}{2\pi i}\int_\delta \frac{f'}{f}(z)\,dz\\ &= 0.\end{aligned}$$

"$\Leftarrow$" Now let $[a_i]$, $[b_j] \in \mathbb{C}/\Lambda$ and $n_i$, $m_j \in \mathbb{N}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ be such that $\sum_{i=n}^n n_i = \sum_{j=m}^m m_j$ and $\sum_{i=1}^n n_i[a_i] = \sum_{j=1}^m m_j[b_j]$. We will construct a meromorphic function $f : \mathbb{C}/\Lambda \to \mathbb{C}$ with zeros $[a_i]$ of order $n_i$ and poles $[b_j]$ of order $m_j$. We choose representatives $a_i$, $b_j \in \mathbb{C}$ for $[a_i]$ resp. $[b_j]$ such that $\sum_{i=1}^n n_i a_i = \sum_{j=1}^m m_j b_j$ and define the function

$$g : \mathbb{C} \to \mathbb{C}; z \mapsto \frac{\prod_{i=1}^n \theta_\sigma(z-a_i)^{n_i}}{\prod_{j=1}^m \theta_\sigma(z-b_j)^{m_j}}$$

where $\theta_\sigma$ is the theta function with characteristic $\frac{1}{2}\tau + \frac{1}{2}$. Obviously $g$ is a meromorphic function with zeros in $a_i + \Lambda$ of order $n_i$ and poles in $b_j + \Lambda$ of order $m_j$. We have to show that $g$ is doubly-periodic with respect to $\Lambda$. Therefore we have to show that $g(z+\lambda) = g(z)$ for all $\lambda \in \Lambda$. It suffices to show that $g(z+1) = g(z)$ and $g(z+\tau) = g(z)$.

$$g(z+1) = \frac{\prod_{i=1}^{n} \theta_\sigma(z+1-a_i)^{n_i}}{\prod_{j=1}^{m} \theta_\sigma(z+1-b_j)^{m_j}} = \frac{\prod_{i=1}^{n} \theta_\sigma(z-a_i)^{n_i}}{\prod_{j=1}^{m} \theta_\sigma(z-b_j)^{m_j}} = g(z)$$

and

$$\begin{aligned} g(z+\tau) &= \frac{\prod_{i=1}^{n} \theta_\sigma(z+\tau-a_i)^{n_i}}{\prod_{j=1}^{m} \theta_\sigma(z+\tau-b_j)^{m_j}} \\ &= \frac{\prod_{i=1}^{n} (e_\sigma(\tau, z-a_i)\theta_\sigma(z-a_i))^{n_i}}{\prod_{j=1}^{m} (e_\sigma(\tau, z-b_j)\theta_\sigma(z-b_j))^{m_j}} \\ &= \frac{\prod_{i=1}^{n} e_\sigma(\tau, z-a_i)^{n_i}}{\prod_{j=1}^{m} e_\sigma(\tau, z-b_j)^{m_j}} \frac{\prod_{i=1}^{n} \theta_\sigma(z-a_i)^{n_i}}{\prod_{j=1}^{m} \theta_\sigma(z-b_j)^{m_j}} \\ &= \frac{\prod_{i=1}^{n} e_\sigma(\tau, z-a_i)^{n_i}}{\prod_{j=1}^{m} e_\sigma(\tau, z-b_j)^{m_j}} \cdot g(z) \end{aligned}$$

but

$$\begin{aligned} \frac{\prod_{i=1}^{n} e_\sigma(\tau, z-a_i)^{n_i}}{\prod_{j=1}^{m} e_\sigma(\tau, z-b_j)^{m_j}} &= \frac{\prod_{i=1}^{n} \exp(-2\pi i(z-a_i+\sigma))^{n_i}}{\prod_{j=1}^{m} \exp(-2\pi i(z-b_j+\sigma))^{m_j}} \\ &= \frac{\prod_{i=1}^{n} \exp(-2\pi i(z+\sigma))^{n_i}}{\prod_{j=1}^{m} \exp(-2\pi i(z+\sigma))^{m_j}} \\ &\qquad \cdot \frac{\prod_{i=1}^{n} \exp(2\pi i a_i)^{n_i}}{\prod_{j=1}^{m} \exp(2\pi i b_j)^{m_j}} \\ &= \frac{\exp(-2\pi i(z+\sigma))^{\sum_{i=1}^{n} n_i}}{\exp(-2\pi i(z+\sigma))^{\sum_{j=1}^{m} m_j}} \\ &\qquad \cdot \frac{\exp(2\pi i \sum_{i=1}^{n} n_i a_i)}{\exp(2\pi i \sum_{j=1}^{m} m_j b_j)} \\ &= 1. \end{aligned}$$

So $g(z+\tau) = g(z)$ as well. Hence $g$ is doubly periodic w.r.t. $\Lambda$ and the function $f : \mathbb{C}/\Lambda \to \mathbb{C}$ with $f([z]) = g(z)$ is well-defined and a solution.

Now suppose we are given two meromorphic functions $f$, $g$ : $\mathbb{C}/\Lambda \to \mathbb{C}$ with zeros $[a_i]$ of order $n_i$ and poles $[b_j]$ of order $m_j$. Then $\frac{f}{g}$ has no zeros or poles. Hence it is constant. □

## 3. WEIERSTRASS $\wp$-FUNCTION

Now we want to capitalize on our work above. Therefore we consider a very special periodic function, the Weierstraß $\wp$-function.

**Definition.** The *Weierstraß* $\wp - \mathit{function}$ is defined to be the function $\wp : \mathbb{C} \to \mathbb{C}$ given by

$$\wp(z) = \frac{1}{z^2} + \sum_{0 \neq \lambda \in \Lambda} \left( \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

**Proposition 8.** (Without proof) *$\wp$ is a $\Lambda$-periodic meromorphic function with poles of order 2 exactly in $\Lambda$.*

The following lemma gives a connection between the Weierstraß $\wp$-function and our well known theta function with characteristic $\sigma = \frac{1}{2} + \frac{1}{2}\tau$.

**Lemma 9.** *There is a constant $c \in \mathbb{C}$ such that*

$$\wp(z) = -\left( \frac{\theta'_\sigma}{\theta_\sigma} \right)'(z) + c$$

**Note.** The quotient $\frac{\theta'_\sigma}{\theta_\sigma}$ isn't doubly-periodic, but the derivative $\left(\frac{\theta'_\sigma}{\theta_\sigma}\right)'$ is doubly-periodic.

To see this consider $\frac{\theta'_\sigma}{\theta_\sigma}(z+\lambda)$ for some $\lambda = p\tau + q \in \Lambda$.

$$\begin{aligned}
\frac{\theta'_\sigma}{\theta_\sigma}(z+\lambda) &\overset{(5)}{=} \frac{(e_\sigma(\lambda,z)\theta_\sigma(z))'}{e_\sigma(\lambda,z)\theta_\sigma(z)} = \frac{e'_\sigma(\lambda,z)\theta_\sigma(z) + e_\sigma(\lambda,z)\theta'_\sigma(z)}{e_\sigma(\lambda,z)\theta_\sigma(z)} \\
&\overset{def}{=} \frac{\exp'(\pi i\lambda - \pi i p^2\tau - 2\pi i p(z+\sigma))\theta_\sigma(z) + e_\sigma(\lambda,z)\theta'_\sigma(z)}{e_\sigma(\lambda,z)\theta_\sigma(z)} \\
&= \frac{-2\pi i p e_\sigma(\lambda,z)\theta_\sigma(z) + e_\sigma(\lambda,z)\theta'_\sigma(z)}{e_\sigma(\lambda,z)\theta_\sigma(z)} \\
&= -2\pi i p + \frac{\theta'_\sigma}{\theta_\sigma}(z) \neq \frac{\theta'_\sigma}{\theta_\sigma}(z)
\end{aligned}$$

as in general $p \neq 0$. From the equation $\frac{\theta'_\sigma}{\theta_\sigma}(z+\lambda) = -2\pi i p + \frac{\theta'_\sigma}{\theta_\sigma}(z)$ above it follows directly that $\left(\frac{\theta'_\sigma}{\theta_\sigma}\right)'$ is doubly-periodic.

*Proof.* We know that $\theta_\sigma$ is holomorphic and has its zeros precisely in the lattice points $\lambda \in \Lambda$. That means that the expansion of $\frac{\theta'_\sigma}{\theta_\sigma}$ in a Laurent series around 0 looks like

$$\frac{\theta'_\sigma}{\theta_\sigma}(z) = a_{-1}\frac{1}{z} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \text{ terms of higher order}$$

for some constants $a_i \in \mathbb{C}$. We can choose a neighborhood $U$ of 0 such that 0 is the only zero of $\theta_\sigma$ in $U$. As 0 is a single zero we know that

$$a_{-1} = \mathrm{Res}_0\left(\frac{\theta'_\sigma}{\theta_\sigma}\right) = \int_\alpha \frac{\theta'_\sigma}{\theta_\sigma}(z)\, dz = 1$$

where $\alpha : [0,1] \to \mathbb{C}; t \mapsto re^{2\pi i t}$ for some suitable $r$. We conclude

$$\frac{\theta'_\sigma}{\theta_\sigma}(z) = \frac{1}{z} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \text{ terms of higher order}$$

and calculate

$$\left(\frac{\theta'_\sigma}{\theta_\sigma}\right)'(z) = -\frac{1}{z^2} + a_1 + 2a_2 z + 3a_3 z^2 + \text{ terms of higher order}$$

If we add $\wp$ and $\left(\frac{\theta'_\sigma}{\theta_\sigma}\right)'$ then we obtain

$$\wp(z) + \left(\frac{\theta'_\sigma}{\theta_\sigma}\right)'(z) = \sum_{0 \neq \lambda \in \Lambda}\left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right) + a_1 + 2a_2 z + 3a_3 z^2 + \ldots$$

From this sum we see directly that $\wp + \left(\frac{\theta'_\sigma}{\theta_\sigma}\right)'$ doesn't have any poles in $U$. Hence $\wp + \left(\frac{\theta'_\sigma}{\theta_\sigma}\right)'$ is holomorphic in a neighborhood of 0 and thus holomorphic everywhere. As it is in addition doubly-periodic (since $\wp$ is as well as $\left(\frac{\theta'_\sigma}{\theta_\sigma}\right)'$ doubly-periodic) we know from our very first lemma that it must be constant. □

The Weierstraß $\wp$-function satisfies a number of equations and differential equation. This feature makes the Weierstraß $\wp$-function to be of interest. The most important differential equation that is satisfied by the Weierstraß $\wp$-function is the following:

**Theorem 10.** *The Weierstraß $\wp$ -function satisfies the differential equation*

$$\wp'(z)^2 = c_3\wp(z)^3 + c_2\wp(z)^2 + c_1\wp(z) + c_0$$

*where the constants*

$$c_3 = 4\ ,\ c_2 = 0\ ,\ c_1 = -60 \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^4} \text{ and } c_0 = -140 \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^6}$$

*depend on the lattice* $\Lambda$.

*Proof.* Consider $\wp(z) - \frac{1}{z^2} = \sum_{0 \neq \lambda \in \Lambda} \left( \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$. This function is holomorphic in a neighborhood of 0. We can expand the summands $\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}$:

$$\begin{aligned} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} &= \frac{1}{\lambda^2} \left( \frac{1}{\left(1 - \frac{z}{\lambda}\right)^2} - 1 \right) \\ &= \frac{1}{\lambda^2} \left( \left( \sum_{n=0}^{\infty} \left( \frac{z}{\lambda} \right)^n \right)^2 - 1 \right) \\ &= \frac{1}{\lambda^2} \left( 2\frac{z}{\lambda} + 3\frac{z^2}{\lambda^2} + 4\frac{z^3}{\lambda^3} + 5\frac{z^4}{\lambda^4} + \dots \right) \\ &= 2\frac{z}{\lambda^3} + 3\frac{z^2}{\lambda^4} + 4\frac{z^3}{\lambda^5} + 5\frac{z^4}{\lambda^6} + \dots \end{aligned}$$

This sum is absolutely convergent for all $z \in \mathbb{C}$ with $|z| < |\lambda|$; in particular in a neighbourhood of 0.
To simplify the big sum from above we define $s_n := \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^{n+2}}$ for $n \in \mathbb{N}$. Note that $s_n = 0$ for all odd $n \in \mathbb{N}$. We obtain

$$\begin{aligned} \wp(z) &= \frac{1}{z^2} + 2s_1 z + 3s_2 z^2 + 4s_3 z^3 + 5s_4 z^4 + \dots \\ &= \frac{1}{z^2} + 3s_2 z^2 + 5s_4 z^4 + 7s_6 z^6 \dots \end{aligned}$$

which is true in a neigborhood of 0. With the constants

$$c_3 = 4\ ,\ c_2 = 0\ ,\ c_1 = -60 \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^4} \text{ and } c_0 = -140 \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^6}$$

we obtain

$$\wp(z) = \frac{1}{z^2} - \frac{c_1}{20} z^2 - \frac{c_0}{28} z^4 + \text{ terms of higher order}$$

hence

$$\wp'(z) = -\frac{2}{z^3} - \frac{c_1}{10} z - \frac{c_0}{7} z^3 + \text{ terms of higher order}$$

$$\wp'(z)^2 = \frac{4}{z^6} + \frac{2c_1}{5}\frac{1}{z^2} + \frac{4c_0}{7} + \text{ terms of higher order}$$

and

$$\wp(z)^3 = \frac{1}{z^6} - \frac{3c_1}{20}\frac{1}{z^2} - \frac{3c_0}{28} + \text{ terms of higher order}$$

Now consider

$$f(z) := \wp'(z)^2 - c_3\wp(z)^3 - c_1\wp(z) - c_0$$

The series of $f$ has only positive powers of $z$. Hence $f$ is holomorphic around 0. Hence it is holomorphic everywhere. And as it is doubly-periodic, it is constant. But the constant part of the series is $\frac{4}{7}c_0 + 4 \cdot \frac{3}{28}c_0 - c_0 = 0$. Hence $f = 0$. □

We will mention one more equation that is satisfied by the Weierstraß $\wp$-function:

**Remark.** Remember that our lattice $\Lambda$ is generated by 1 and $\tau$. Hence the set of zeros of $\wp'$ is given by $\left(\frac{1}{2} + \Lambda\right) \cup \left(\frac{\tau}{2} + \Lambda\right) \cup \left(\frac{1+\tau}{2} + \Lambda\right)$. Set $e_1 := \wp\left(\frac{1}{2}\right)$, $e_2 := \wp\left(\frac{\tau}{2}\right)$, $e_3 := \wp\left(\frac{1+\tau}{2}\right) \in \mathbb{C}$. Then we have

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$$

and

$$\begin{aligned} e_1 + e_2 + e_3 &= 0 \\ e_1e_2 + e_1e_3 + e_2e_3 &= \frac{1}{4}c_1 \\ e_1e_2e_3 &= -\frac{1}{4}c_0 \end{aligned}$$

where $c_0$ and $c_1$ are the constants from above.

Finally we will see how to use the Weierstraß $\wp$-function to give a group structure to an elliptic curve.

**Remark.** If we consider the elliptic curve

$$C := \{(x, y) \in \mathbb{C}^2 \text{ such that } y^2 = c_3x^3 + c_2x^2 + c_1x + c_0\}$$

for the constants

$$c_3 = 4 \ , \ c_2 = 0 \ , \ c_1 = -60 \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^4} \text{ and } c_0 = -140 \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^6}$$

from the theorem above then we have a bijection

$$\mathbb{C}/\Lambda \setminus \{0\} \to C \text{ given by } z \mapsto (\wp(z), \wp'(z))$$

In particular we can give the variety $C$ the group structure of $\mathbb{C}/\Lambda$.

This can be extended to an embedding of $\mathbb{C}/\Lambda$ into the projective plane. For more details see the article of M. Khalid [2].

## REFERENCES

[1] A. Gathmann, *Algebraic Geometry*, Notes for a class taught at the University of Kaiserslautern 2002/2003, available at `http://www.mathematik.uni-kl.de/ gathmann/de/pub.html`

[2] M. Khalid, *Group Law on the Cubic Curve*, this issue.

[3] D. Mumford, *Tata Lectures on Theta*, Progress in Mathematics Vol 28, Birkhauser Verlag, 1983.

[4] G. Trautmann, *Complex Analysis II*, Notes for a class taught at the University of Kaiserslautern 1996/1997.

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