The Centre of Unitary Isotopes of JB^*-Algebras

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Abstract. We identify the centre of unitary isotopes of a JB^*-algebra. We show that the centres of any two unitary isotopes of a JB^*-algebra are isometrically Jordan *-isomorphic to each other. However, there need be no inclusion between centres of the two unitary isotopes.

1. Basics

We begin by recalling (from [3], for instance) the following concepts of homotope and isotope of Jordan algebras.

Let $\mathcal{J}$ be a Jordan algebra, cf. [3], and $x \in \mathcal{J}$. The x-homotope of $\mathcal{J}$, denoted by $\mathcal{J}_{[x]}$, is the Jordan algebra consisting of the same elements and linear algebra structure as $\mathcal{J}$ but a different product, denoted by "$\cdot_x$", defined by

$$a \cdot_x b = \{axb\}$$

for all a, b in $\mathcal{J}_{[x]}$. By $\{pqr\}$ we will always denote the Jordan triple product of p, q, r defined in the Jordan algebra $\mathcal{J}$ as below:

$$\{pqr\} = (p \circ q) \circ r - (p \circ r) \circ q + (q \circ r) \circ p\,,$$

where $\circ$ stands for the original Jordan product in $\mathcal{J}$. An element x of a Jordan algebra $\mathcal{J}$ with unit e is said to be invertible if there exists $x^{-1} \in \mathcal{J}$, called the inverse of x, such that $x \circ x^{-1} = e$ and $x^2 \circ x^{-1} = x$. The set of all invertible elements of $\mathcal{J}$ will be denoted by $\mathcal{J}_{inv}$. In this case, x acts as the unit for the homotope $\mathcal{J}_{[x^{-1}]}$ of $\mathcal{J}$.

If $\mathcal{J}$ is a unital Jordan algebra and $x \in \mathcal{J}_{inv}$ then by x-isotope of $\mathcal{J}$, denoted by $\mathcal{J}^{[x]}$, we mean the x^{-1}-homotope $\mathcal{J}_{[x^{-1}]}$ of $\mathcal{J}$. We denote the multiplication "$\cdot_{x^{-1}}$" of $\mathcal{J}^{[x]}$ by "$\circ_x$".

The following lemma gives the invariance of the set of invertible elements in a unital Jordan algebra on passage to any of its isotopes.

Lemma 1.1. *For any invertible element a in a unital Jordan algebra $\mathcal{J}$, $\mathcal{J}_{inv} = \mathcal{J}_{inv}^{[a]}$.*

Proof. See Lemma 1.5 of [8]. □

Let $\mathcal{J}$ be a Jordan algebra and let $a, b \in \mathcal{J}$. The operators T_b and $U_{a,b}$ are defined on $\mathcal{J}$ by $T_b(x) = b \circ x$ and $U_{a,b}(x) = \{axb\}$. We shall denote $U_{a,a}$ simply by U_a. The elements a and b are said to operator commute if T_a commute with T_b.

Let $\mathcal{J}$ be a complex unital Banach Jordan algebra and let $x \in \mathcal{J}$. As usual, the spectrum of x in $\mathcal{J}$, denoted by $\sigma_{\mathcal{J}}(x)$, is defined by

$$\sigma_{\mathcal{J}}(x) = \{\lambda \notin \mathbb{C} : x - \lambda e \text{ is not invertible in } \mathcal{J}\} .$$

A Jordan algebra $\mathcal{J}$ with product $\circ$ is called a Banach Jordan algebra if there is a norm $\|.\|$ on $\mathcal{J}$ such that $(\mathcal{J}, \|.\|)$ is a Banach space and $\|a \circ b\| \leq \|a\| \, \|b\|$. If, in addition, $\mathcal{J}$ has a unit e with $\|e\| = 1$ then $\mathcal{J}$ is called a unital Banach Jordan algebra. In the sequel, we will only be considering unital Banach Jordan algebras; the norm closure of the Jordan subalgebra $J(x_1, \ldots, x_r)$ generated by $x_1, \ldots, x_r$ of Banach Jordan algebra $\mathcal{J}$ will be denoted by $\mathcal{J}(x_1, \ldots, x_r)$.

The following elementary properties of Banach Jordan algebras are similar to those of Banach algebras and their proofs are a fairly routine modifications of these [1, 2, 7, 9].

Lemma 1.2. *Let $\mathcal{J}$ be a Banach Jordan algebra with unit e and $x_1, \ldots, x_r \in \mathcal{J}$.*

(i) *If $J(x_1, \ldots, x_r)$ is an associative subalgebra of $\mathcal{J}$, then $\mathcal{J}(x_1, \ldots, x_r)$ is a commutative Banach algebra.*
(ii) *T_{x_1} and U_{x_1,x_2} are continuous with $\|T_{x_1}\| \leq \|x_1\|$ and $\|U_{x_1,x_2}\| \leq 3\|x_1\| \, \|x_2\|$.*
(iii) *$\mathcal{J}(x_1, \ldots, x_r)$ is a closed subalgebra of $\mathcal{J}$.*
(iv) *If $\mathcal{J}$ is unital then $\mathcal{J}(e, x_1)$ is a commutative Banach algebra.*
(v) *If $x \in \mathcal{J}$ and $\|x\| < 1$ then $e - x$ is invertible and $(e-x)^{-1} = \sum_{n=0}^{\infty} x^n \in \mathcal{J}(e, x)$.*
(vi) *If K is a closed Jordan subalgebra of $\mathcal{J}$ containing e and $x \in K$ such that $\mathbb{C} \setminus \sigma_{\mathcal{J}}(x)$ is connected then $\sigma_{\mathcal{J}}(x) = \sigma_K(x)$.*

We are interested in a special class of Banach Jordan algebras, called JB^*-algebras. These include all C^*-algebras as a proper subclass (see [10, 13]).

A complex Banach Jordan algebra $\mathcal{J}$ with isometric involution * (see [6], for instance) is called a *JB^*-algebra* if $\|\{xx^*x\}\| = \|x\|^3$ for all $x \in \mathcal{J}$.

The class of JB^*-algebras was introduced by Kaplansky in 1976 (see [10]) around the same time when a related class called JB-algebras was being studied by Alfsen, Shultz and Størmer (see [1]).

A real Banach Jordan algebra $\mathcal{J}$ is called a *JB-algebra* if $\|x\|^2 = \|x^2\| \leq \|x^2 + y^2\|$ for all $x, y \in \mathcal{J}$.

These two classes of algebras are linked as follows (see [10, 13]).

Theorem 1.3. (a) *If $\mathcal{A}$ is a JB^*-algebra then the set of self-adjoint elements of $\mathcal{A}$ is a JB-algebra.*

(b) *If $\mathcal{B}$ is a JB-algebra then under a suitable norm the complexification $\mathcal{C}_{\mathcal{B}}$ of $\mathcal{B}$ is a JB^*-algebra.*

There is an easier subclass of these algebras. Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the full algebra of bounded linear operators on $\mathcal{H}$.

(a) Any closed self-adjoint complex Jordan subalgebra of $\mathcal{B}(\mathcal{H})$ is called a *JC^*-algebra.*

(b) Any closed real Jordan subalgebra of self-adjoint operators of $\mathcal{B}(\mathcal{H})$ is called a *JC-algebra.*

Any JB^*-algebra isometrically *-isomorphic to a JC^*-algebra is also called a JC^*-algebra; similarly, any JB-algebra isometrically isomorphic to a JC-algebra is also called a JC-algebra.

It is easy to verify that a JC^*-algebra is a JB^*-algebra and a JC-algebra is a JB-algebra. It might be expected, conversely, that every JB-algebra is a JC-algebra (with a corresponding statement for JB^*-algebras and JC^*-algebras) but unfortunately this is not true (for details see [1]).

2. UNITARY ISOTOPES OF A JB^*-ALGEBRA

In [8], we presented a study of unitary isotopes of JB^*-algebras. In this section, we recall some facts from [8] which are needed for the sequel.

Let $\mathcal{J}$ be a JB^*-algebra. The element $u \in \mathcal{J}$ is called *unitary* if $u^* = u^{-1}$, the inverse of u. The set of all unitary elements of $\mathcal{J}$

will be denoted by $\mathcal{U}(\mathcal{J})$. If u is a unitary element of JB^*-algebra $\mathcal{J}$ then the isotope $\mathcal{J}^{[u]}$ is called a unitary isotope of $\mathcal{J}$.

Theorem 2.1. *Let u be a unitary element of the JB^*-algebra $\mathcal{J}$. Then the isotope $\mathcal{J}^{[u]}$ is a JB^*-algebra having u as its unit with respect to the original norm and the involution $*_u$ defined by $x^{*_u} = \{ux^*u\}$.*

Proof. See Theorem 2.4 of [8]. □

Recall (from [3], for instance) that a Jordan algebra is said to be *special* if it is isomorphic to a Jordan subalgebra of some associative algebra. We require the following fact.

Lemma 2.2. *If $\mathcal{J}$ is a special Jordan algebra and $a \in \mathcal{J}$, then $\mathcal{J}_{[a]}$ is a special Jordan algebra.*

Proof. See Lemma 1.3 in [8]. □

Theorem 2.3. *The unitary isotope of a JC^*-algebra is again a JC^*-algebra.*

Proof. This follows from Theorem 2.1 and Lemma 2.2 (also see [8, Theorem 2.12]). □

We close this section by noting following facts.

Lemma 2.4. *Let $\mathcal{J}$ be a JB^*-algebra with unit e. Then $u \in \mathcal{U}(\mathcal{J}) \Longrightarrow e \in \mathcal{U}(\mathcal{J}^{[u]})$. Moreover $\mathcal{J}^{[u]^{[e]}} = \mathcal{J}$.*

Proof. See Lemma 2.7 of [8]. □

Next theorem establishes the invariance of unitaries on passage to unitary isotopes of a JB^*-algebra.

Theorem 2.5. *For any unitary element u in the JB^*-algebra $\mathcal{J}$,*

$$\mathcal{U}(\mathcal{J}) = \mathcal{U}(\mathcal{J}^{[u]}) .$$

Proof. See Theorem 2.8 of [8]. □

Corollary 2.6. *Let $\mathcal{J}$ be a JB^*-algebra with unit e and let $u, v \in \mathcal{U}(\mathcal{J})$. Then*

(i) $\mathcal{J}^{[u]^{[v]}} = \mathcal{J}^{[v]}$.

(ii) *The relation of being unitary isotope is an equivalence relation in the class of unital JB^*-algebras.*

Proof. See Corollary 2.9 of [8]. □

3. Centre of Unitary Isotopes

In this section, we identify the centre of unitary isotopes in terms of the centre of the original JB^*-algebra. We recall the following definition from [14].

Definition 3.1. Let $\mathcal{J}$ be a unital JB^*-algebra and let

$$C(\mathcal{J}) = \{x \in \mathcal{J}_{sa} : x \text{ operator commutes with every y} \in \mathcal{J}_{sa}\}\ .$$

Then the *centre* of $\mathcal{J}$, denoted by $\mathcal{Z}(\mathcal{J})$, is defined by

$$\mathcal{Z}(\mathcal{J}) = C(\mathcal{J}) + iC(\mathcal{J})\ .$$

Remark 3.2. It is known from [14] that $\mathcal{Z}(\mathcal{J})$ is a C^*-algebra, and if $\mathcal{J}$ is a JC^*-algebra with $\mathcal{J} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ then

$$\mathcal{Z}(\mathcal{J}) = \{x \in \mathcal{J} : xy = yx \quad \forall y \in \mathcal{J}\}\ .$$

To investigate further properties of the centre we need the following lemma.

Lemma 3.3. *Let $\mathcal{J}$ be a JB^*-algebra and let $x \in \mathcal{Z}(\mathcal{J})$. Then for all $y \in \mathcal{J}$,*

(i) $T_xT_y = T_yT_x$;
(ii) $T_xU_y = U_yT_x$;
(iii) $U_xU_y = U_yU_x$;
(iv) *if $u \in \mathcal{J}$ is unitary then $(x \circ u^*) \circ u = x$.*

Proof. Let $x = a + ib$ and $y = c + id$ with $a, b \in C(\mathcal{J})$ and $c, d \in \mathcal{J}_{sa}$. Then

$$\begin{aligned} T_xT_y &= (T_a + iT_b)(T_c + iT_d) = T_aT_c + iT_aT_d + iT_bT_c - T_bT_d \\ &= T_cT_a + iT_dT_a + iT_cT_b - T_dT_b = T_yT_x \end{aligned}$$

as $a, b \in C(\mathcal{J})$ which proves (i).
(ii). Since $U_y = 2T_y^2 - T_{y^2}$, we have

$$T_xU_y = T_x(2T_y^2 - T_{y^2}) = 2T_xT_y^2 - T_xT_{y^2} = (2T_y^2 - T_{y^2})T_x = U_yT_x$$

by part (i) (note that the associativity of $\mathcal{B}(\mathcal{J})$ is used here).
(iii). Since $x \in \mathcal{Z}(\mathcal{J})$, $x^2 \in \mathcal{Z}(\mathcal{J})$ by Remark 3.2. Hence by part (ii),

$$\begin{aligned} U_xU_y &= (2T_x^2 - T_{x^2})U_y = 2T_x^2U_y - T_{x^2}U_y \\ &= 2U_yT_x^2 - U_yT_{x^2} = U_yU_x. \end{aligned}$$

(iv). By part (i), $(x \circ u^*) \circ u = T_uT_xu^* = T_xT_uu^* = T_xe = x$. □

Theorem 3.4. *Let $\mathcal{J}$ be a JB^*-algebra with unit e and let $b \in \mathcal{Z}(\mathcal{J})$. Then for any unitary $u \in \mathcal{U}(\mathcal{J})$ and for any $x \in \mathcal{J}$ we have*

(i) $(u^* \circ x) \circ u = u^* \circ (x \circ u)$;

(ii) $\{(b \circ u)u^* x\} = b \circ x$.

Proof. (i). If $\mathcal{J}$ is special then

$$\begin{aligned}(u^* \circ x) \circ u &= \frac{1}{4}(u(u^*x + xu^*) + (u^*x + xu^*)u) \\ &= \frac{1}{4}(2x + uxu^* + u^*xu) \\ &= \frac{1}{4}(u^*(ux + xu) + (ux + xu)u^*) = u^* \circ (x \circ u).\end{aligned}$$

Hence, by the *Shirshov–Cohn theorem with inverses* [5], we have in the general case $(u^* \circ x) \circ u = u^* \circ (x \circ u)$.

(ii). Since $b \in \mathcal{Z}(\mathcal{J})$ and $u \in \mathcal{U}(\mathcal{J})$, we get by Lemma 3.3 (iv) that

$$(b \circ u) \circ u^* = b. \tag{1}$$

Again by Lemma 3.3 (i),

$$(u^* \circ x) \circ (b \circ u) = T_{(u^* \circ x)} T_b u = T_b T_{(u^* \circ x)} u = b \circ (u \circ (x \circ u^*)) ,$$

and

$$u^* \circ ((b \circ u) \circ x) = T_{u^*} T_x T_b u = T_b T_{u^*} T_x u = b \circ (u^* \circ (x \circ u)) ,$$

so by part (i)

$$(u^* \circ x) \circ (b \circ u) = u^* \circ ((b \circ u) \circ x). \tag{2}$$

Thus by (1) and (2),

$$\begin{aligned}\{(b \circ u)u^* x\} &= ((b \circ u) \circ u^*) \circ x + (u^* \circ x) \circ (b \circ u) - ((b \circ u) \circ x) \circ u^* \\ &= b \circ x .\end{aligned}$$

□

We now need a characterisation of the centre in terms of Hermitian operators. These are defined in terms of the numerical range of operators as follows (see [14], for example).

Definition 3.5. If $\mathcal{J}$ is a complex unital Banach Jordan algebra with unit e and $D(\mathcal{J}) = \{f \in \mathcal{J}^* : f(e) = \|f\| = 1\}$ then, for $a \in \mathcal{J}$, the *numerical range* of a, denoted by $W(a)$, is defined by $W(a) = \{f(a) : f \in D(\mathcal{J})\}$. The element a is called *Hermitian* if $W(a) \subseteq \mathbb{R}$. The set of all Hermitian elements of $\mathcal{J}$ is denoted by $Her\mathcal{J}$.

The Hermitian elements in a unital JB^*-algebra are exactly the self-adjoint elements (see [13]) but we shall need the following characterisation of the Hermitian operators on a JB^*-algebra, given in [14].

Theorem 3.6. *Let $\mathcal{J}$ be a JB^*-algebra with unit e. Then $S \in Her\,\mathcal{B}(\mathcal{J})$ if and only if $S = T_a + \delta$ where δ is a *-derivation and $a = S(e)$ is self-adjoint.*

We can now give a characterisation of the centre of a unitary isotope.

Theorem 3.7. *Let $\mathcal{J}$ be a JB^*-algebra with unit e and let $u \in \mathcal{U}(\mathcal{J})$. Let $\mathcal{A}$ be a JC^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ with unit e_A and let $w \in \mathcal{U}(\mathcal{A})$.*

(i) *If $x \in \mathcal{Z}(\mathcal{J})$ then $u \circ x \in \mathcal{Z}(\mathcal{J}^{[u]})$.*

(ii) *If $a \in \mathcal{Z}(\mathcal{A}^{[w]})$ then $(a \circ w^*) \circ w = a$.*

(iii) *If $z \in \mathcal{Z}(\mathcal{J}^{[u]})$ then $u \circ (u^* \circ z) = z$.*

(iv) *Define $\psi : \mathcal{Z}(\mathcal{J}) \to \mathcal{Z}(\mathcal{J}^{[u]})$ by $\psi(x) = u \circ x$. Then ψ is an isometric *-isomorphism of $\mathcal{Z}(\mathcal{J})$ onto $\mathcal{Z}(\mathcal{J}^{[u]})$.*

Proof. (i). Let $x = a + ib$ where $a, b \in \mathcal{Z}(\mathcal{J})_{sa}$. Let $S = T_a \in Her\,\mathcal{B}(\mathcal{J})$. Then

$$S(e) = T_a(e) = a \circ e = a \quad \text{and} \quad S(u) = u \circ a\,.$$

As $S \in Her\,\mathcal{B}(\mathcal{J})$, $S(u) \in (\mathcal{J}^{[u]})_{sa}$ by Theorem 3.6. By Theorem 3.4 (ii),

$$S(y) = T_a(y) = a \circ y = \{(a \circ u)u^* y\} = (a \circ u) \circ_u y$$

for all $y \in \mathcal{J}$. Therefore, $S(y) = L^{[u]}_{S(u)}(y)$ for all $y \in \mathcal{J}$, where operator $L^{[u]}_{S(u)}$ stands for the multiplication by $S(u)$ in $\mathcal{J}^{[u]}$. Moreover, as $a \in \mathcal{Z}(\mathcal{J})$ we get by [14, Theorem 14] that $S^2 \in Her\,\mathcal{B}(\mathcal{J}) = Her\,\mathcal{B}(\mathcal{J}^{[u]})$ because $\mathcal{B}(\mathcal{J}^{[u]}) = \mathcal{B}(\mathcal{J})$ (see Theorem 2.1). So again by [14, Theorem 14], $S(u) \in \mathcal{Z}(\mathcal{J}^{[u]})$ as $S = L^{[u]}_{S(u)}$. Therefore, $u \circ a \in \mathcal{Z}(\mathcal{J}^{[u]})_{sa}$. Similarly, $u \circ b \in \mathcal{Z}(\mathcal{J}^{[u]})_{sa}$. Hence $u \circ x = u \circ a + iu \circ b \in \mathcal{Z}(\mathcal{J}^{[u]})$.

(ii). By Remark 3.2,

$$\mathcal{Z}(\mathcal{A}) = \{x \in \mathcal{A} : xy = yx\}. \tag{3}$$

By Theorem 2.3, the isotope $\mathcal{A}^{[w]}$ is a JC^*-algebra and

$$\mathcal{Z}(\mathcal{A}^{[w]}) = \{x \in \mathcal{A} : xw^*y = yw^*x\}. \tag{4}$$

Now, if $a \in \mathcal{Z}(\mathcal{A}^{[w]})$ then (by (4)) $aw^*y = yw^*a$ for all $y \in \mathcal{A}$. In particular,

$$aw^* = w^*a. \tag{5}$$

By part (i), $a \circ w^* = e_A \circ_w a \in \mathcal{Z}(\mathcal{A}^{[w]^{[e_A]}}) = \mathcal{Z}(\mathcal{A})$. So we have by (4) that

$$(a \circ w^*) \circ w = (a \circ w^*)w = \frac{1}{2}(aw^* + w^*a)w$$

hence by (5)

$$(a \circ w^*) \circ w = (aw^*)w = a(w^*w) = a\,,$$

as required.

(iii) Now, let v be any unitary in $\mathcal{Z}(\mathcal{J}^{[u]})$ (the centre of the unitary isotope $\mathcal{J}^{[u]}$ of the JB^*-algebra $\mathcal{J}$). Then v is a unitary in $\mathcal{J}$ by Theorem 2.5. By [8, Corollary 1.14], $\mathcal{J}(e, u, u^*, v, v^*)$ is a JC^*-algebra and $v \in \mathcal{Z}((\mathcal{J}(e, u, u^*, v, v^*))^{[u]})$. Hence, by (ii),

$$u \circ (u^* \circ v) = v. \tag{6}$$

If $z \in \mathcal{Z}(\mathcal{J}^{[u]})$, then by the Russo–Dye Theorem (cf. [11]) for C^*-algebras there exist unitaries $v_j \in \mathcal{Z}(\mathcal{J}^{[u]})$ and scalars $0 \leq \lambda_j \leq 1$ with $\sum_{j=1}^{n} \lambda_j = 1$ for some $n \in \mathcal{N}$ such that $\frac{z}{\|z\|+1} = \sum_{j=1}^{n} \lambda_j v_j$ because $\|\frac{z}{\|z\|+1}\| < 1$ (recall that $\mathcal{Z}(\mathcal{J}^{[u]})$ is a C^*-algebra). Hence, by (6),

$$\begin{aligned} u \circ (u^* \circ z) &= u \circ (u^* \circ (\|z\| + 1) \sum_{j=1}^{n} \lambda_j v_j\,) \\ &= (\|z\| + 1) \sum_{j=1}^{n} \lambda_j (u \circ (u^* \circ v_j)) \\ &= (\|z\| + 1) \sum_{j=1}^{n} \lambda_j v_j = z. \end{aligned}$$

(iv). As $\psi = T_u\,|_{\mathcal{Z}(\mathcal{J})}$, ψ is linear and continuous by Lemma 1.2 (i). Let $z \in \mathcal{Z}(\mathcal{J}^{[u]})$. Applying part (i) to $\mathcal{J}^{[u]}$ we get $e \circ_u z \in \mathcal{Z}(\mathcal{J}^{[u]^{[e]}})$. But $\mathcal{J}^{[u]^{[e]}} = \mathcal{J}$ by Lemma 2.4 and $e \circ_u z = \{eu^*z\} = u^* \circ z$. Hence $u^* \circ z \in \mathcal{Z}(\mathcal{J})$. Moreover, $\psi(u^* \circ z) = u \circ (u^* \circ z) = z$ by part (iii). Thus ψ maps $\mathcal{Z}(\mathcal{J})$ onto $\mathcal{Z}(\mathcal{J}^{[u]})$.

Further, $\|\psi(x)\| \leq \|u\| \|x\|$ while, by Lemmas 3.3 (i) and 1.2 (ii),

$$\|x\| = \|T_x T_{u^*} u\| = \|T_{u^*} T_x u\| \leq \|x \circ u\| = \|\psi(x)\| \ .$$

Thus ψ is an isometry.

Finally, as $\psi(e) = u$ and u is the unit of $\mathcal{J}^{[u]}$ it follows from [12, Theorem 6] that ψ is an isometric *-isomorphism. □

Corollary 3.8. *Let $\mathcal{J}$ be a unital JB^*-algebra. Then, for all $u, v \in \mathcal{U}(\mathcal{J})$, $\mathcal{Z}(\mathcal{J}^{[u]})$ is isometrically Jordan *-isomorphic to $\mathcal{Z}(\mathcal{J}^{[v]})$.*

Proof. By Theorem 2.5, $v \in \mathcal{U}(\mathcal{J})$. Hence, by Theorem 3.7, $\mathcal{Z}(\mathcal{J}^{[u]})$ is isometrically *-isomorphic to $\mathcal{Z}(\mathcal{J}^{[u]^{[v]}})$. However, by Corollary 2.6 (i), $\mathcal{J}^{[u]^{[v]}} = \mathcal{J}^{[v]}$. This gives the required result. □

An alternative proof of above Corollary 3.8 can be obtained by noting that $\mathcal{Z}(\mathcal{J}^{[u]})$ is isometrically *-isommorphic to $\mathcal{Z}(\mathcal{J})$ and $\mathcal{Z}(\mathcal{J})$ is isometrically *-isomorphic to $\mathcal{Z}(\mathcal{J}^{[v]})$ by Theorem 3.7 (applied twice). As the next example shows there need be no inclusion between the centre of a unital JB^*-algebra and the centre of its isotopes. In the following discussion $\mathcal{M}_2(\mathbb{C})$ denotes the standard complexification of the real Jordan algebra of all 2×2 symmetric matrices.

Example 3.9. If $u \in \mathcal{U}(\mathcal{M}_2(\mathbb{C})) \setminus \mathcal{Z}(\mathcal{M}_2(\mathbb{C}))$ then the unit $e \notin \mathcal{Z}(\mathcal{M}_2(\mathbb{C})^{[u]})$.

Indeed, $\mathcal{M}_2(\mathbb{C})^{[u]}$ is a 4-dimensional C^*-algebra by Theorem 2.3 with 1-dimensional centre by the above Theorem 3.7. As u does not belong to $\mathcal{Z}(\mathcal{M}_2(\mathbb{C}))$, $u \notin Sp(e)$ where $Sp(e)$ denotes the linear span of e, and hence $e \notin Sp(u)$. This gives that $e \notin \mathcal{Z}(\mathcal{M}_2(\mathbb{C})^{[u]})$.

As a final point on the relationships between the centres it should be noted in the proof of Theorem 3.7 (i) that if $a \in \mathcal{Z}(\mathcal{J})$ and $S = T_a$ then S is left multiplication in any unitary isotope. In order to study the *-derivations it might be hoped that if $T \in Her\,\mathcal{B}(\mathcal{J})$ then there exists a unitary isotope $\mathcal{J}^{[u]}$ such that T is left multiplication operator in $Her\,\mathcal{B}(\mathcal{J}^{[u]})$ since as linear spaces $\mathcal{B}(\mathcal{J}) = \mathcal{B}(\mathcal{J}^{[u]})$ so $Her\,\mathcal{B}(\mathcal{J}) = Her\,\mathcal{B}(\mathcal{J}^{[u]})$. Unfortunately, this fails even when $\mathcal{J} = \mathcal{M}_2(\mathbb{C})$. As all *-derivations are inner in this case, it follows that $T \in Her\,\mathcal{B}(\mathcal{M}_2(\mathbb{C}))$ if and only if $T = l_a + r_b$ where $a, b \in (\mathcal{M}_2(\mathbb{C}))_{sa}$ and $l_a(x) = ax$ and $r_b(x) = xb$.

Corollary 3.10. *If* $a, b \in \mathcal{M}_2(\mathbb{C})$ *are given by* $a = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 6 & 0 \\ 0 & 23 \end{pmatrix}$ *and* $T \in Her\,\mathcal{B}(\mathcal{M}_2(\mathbb{C}))$ *is defined by* $T = l_a + r_b$, *then* T *is not left multiplication in any unitary isotope.*

Proof. It was noted in Example 3.9 that if $u \in \mathcal{U}(\mathcal{M}_2(\mathbb{C}))$ then $\mathcal{M}_2(\mathbb{C})^{[u]}$ is a four-dimensional C^*-algebra with a one-dimensional centre so is isomorphic to $\mathcal{M}_2(\mathbb{C})$. By [4, Theorem 10], $\sigma(T) = \sigma(a) + \sigma(b) = \{7, 8, 24, 25\}$.

On the other hand, if $L_c^{[u]} \in Her\,\mathcal{B}(\mathcal{M}_2(\mathbb{C}))$ with say $\sigma_{\mathcal{M}_2(\mathbb{C})}(c) = \{\lambda_1, \lambda_2\}$ then $\sigma(L_c^{[u]}) = \{\lambda_1, \frac{\lambda_1+\lambda_2}{2}, \lambda_2\}$ again by [4, Theorem 10], so $\sigma(L_c^{[u]})$ contains only three points. Hence $\sigma(T) \neq \sigma(L_c^{[u]})$ for any unitary $u \in \mathcal{U}(\mathcal{M}_2(\mathbb{C}))$. □

Acknowledgement. The author is indebted to Dr Martin A. Youngson for his help and encouragement during this work.

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Received on 26 September 2006 and in revised form on 9 May 2007.