Some Mean Inequalities

FINBARR HOLLAND

Dedicated to Trevor West on the occasion of his retirement.

ABSTRACT. Let $\mathbb{P}$ denote the collection of positive sequences defined on the set of natural numbers $\mathbb{N}$. It is proved that if $x \in \mathbb{P}$, and $s < 0$, then

$$\frac{1}{n}\sum_{k=1}^{n}\Big(\frac{1}{k}\sum_{j=1}^{k} x_j^{1/s}\Big)^s \leq \Big(\frac{1}{n}\sum_{j=1}^{n}\Big(\frac{1}{j}\sum_{k=1}^{j} x_k\Big)^{1/s}\Big)^s, \quad n \in \mathbb{N}$$

with equality if and only if x is a constant sequence. This is a sharp refinement of an inequality discovered by Knopp in 1928.

1. Introduction

When I received the invitation to participate in the *Westfest*, I was in the throes of writing up a solution to the following problem, due to Joel Zinn, which was posed in the American Mathematical Monthly, and I offered to speak on this topic at the meeting in TCD to mark Trevor's retirement. I'm grateful to the organising committee of the *Westfest* for giving me the opportunity to do so.

Problem 1 (Number 11145). *Find the least c such that if $n \geq 1$, $a_1, \dots, a_n > 0$, then*

$$\sum_{k=1}^{n} \frac{k}{\sum_{j=1}^{k} \frac{1}{a_j}} \leq c \sum_{k=1}^{n} a_k.$$

I propose to describe a method to handle a family of similar problems of which this, and classical ones due to Carleman, and Knopp, are special cases.

2. BACKGROUND

We denote by $\mathbb{P}$ the collection of positive sequences $x : \mathbb{N} \to (0, \infty)$. Clearly, $\mathbb{P}$ is a convex set. It is closed under the usual pointwise operations of addition and multiplication, and ordered by the relation:

$$x \le y \iff x_n \le y_n, \ \forall n \in \mathbb{N}.$$

In particular, $\mathbb{P}$ is a commutative group under multiplication, with the sequence vector e of ones acting as the identity. We'll write $1/x$ for the multiplicative inverse of $x \in \mathbb{P}$:

$$(1/x)_n = \frac{1}{x_n}, \ \forall n \in \mathbb{N}.$$

We recall a number of familiar functions that take $\mathbb{P}$ into itself:

$$A : \mathbb{P} \to \mathbb{P}; \ \ A(x)_n = \frac{1}{n} \sum_{k=1}^{n} x_k, \ n = 1, 2, \ldots;$$

$$G : \mathbb{P} \to \mathbb{P}; \ \ G(x)_n = \sqrt[n]{\prod_{k=1}^{n} x_k}, \ n = 1, 2, \ldots;$$

$$H : \mathbb{P} \to \mathbb{P}; \ \ H(x)_n = \frac{n}{\sum_{k=1}^{n} \frac{1}{x_k}}, \ n = 1, 2, \ldots;$$

$$\min : \mathbb{P} \to \mathbb{P}; \ \ \min(x)_n = \min\{x_k : k = 1, 2, \ldots, n\}.$$

These are, respectively, the arithmetic, geometric, harmonic and minimum means. (Weighted versions of these exist, but I'll not have any need to refer to them.)

It is a well-known fact [5] that

$$\min(x)_n \le \frac{n}{\sum_{k=1}^{n} \frac{1}{x_k}} \le \sqrt[n]{\prod_{k=1}^{n} x_k} \le \frac{1}{n} \sum_{k=1}^{n} x_k, \ n = 1, 2, \ldots.$$

Moreover, the inequalities are strict unless x is a constant sequence. Equivalently,

$$\min \le H \le G \le A.$$

It's clear from the definitions that A, G, H and min are "homogeneous" in the sense that, if $f \in \{A, G, H, \min\}$, then

$$f(\lambda x) = \lambda f(x), \quad \forall x \in \mathbb{P}, \lambda > 0.$$

It's perhaps less obvious, but nonetheless true, that they are *super-additive*: if $f \in \{A, G, H, \min\}$, then

$$f(x) + f(y) \leq f(x+y), \ \forall x, y \in \mathbb{P}.$$

Hence they are also concave on $\mathbb{P}$.

We also introduce a one-parameter family of functions $\{M_t : t > 0\}$ that leave $\mathbb{P}$ invariant. If $x \in \mathbb{P}$, we define $M_t(x)$ by

$$M_t(x)_n = \left(\frac{n}{\sum_{j=1}^{n} \frac{1}{x_j^{1/t}}} \right)^t, \ n = 1, 2, \ldots,$$

so that $M_t(x) = (H(x^{1/t}))^t$,

$$\min(x) \leq M_t(x) \leq G(x) \leq A(x), \forall x \in \mathbb{P}, \ \forall t > 0,$$

and

$$\lim_{t \to 0^+} M_t(x) = \min(x), \ \lim_{t \to \infty} M_t(x) = G(x), \forall x \in \mathbb{P}.$$

3. An Inequality Between Compositions of Means

I'm interested in compositional relationships between these various functions. I'll describe the following result.

Theorem 1. *Let $t > 0$. Then $A \circ M_t \leq M_t \circ A$. Moreover, $A \circ M_t(x) = M_t \circ A(x)$ if and only if $x = \lambda e$ for some $\lambda > 0$.*

For instance, when $t = 1$, the claim is that $A \circ H \leq H \circ A$. Equivalently,

$$\frac{1}{n} \sum_{k=1}^{n} \frac{k}{\sum_{j=1}^{k} \frac{1}{x_j}} \leq \frac{n}{\sum_{k=1}^{n} \frac{k}{\sum_{j=1}^{k} x_j}}, \ n = 1, 2, \ldots$$

Even for small values of n this is already fairly challenging, as the reader may discover for him or her self by considering the special case $n = 3$.

A more general weighted inequality of this kind was first postulated by Nanjundiah [13] in 1952, but he offered no proof, and indeed his conjecture is not true generally. A special case of it was conjectured by myself [6] in 1992, and Kedlaya [8] supplied a proof of this in 1994, namely that $A \circ G \leq G \circ A$. In 1996, Mond and Pečarić [12] proved an analogue of the inequality $A \circ H \leq H \circ A$, the case $t = 1$, for Hermitian matrices.

To establish the theorem, we begin by proving a lemma.

Lemma 1. *Let $t > 0$ and let $p = t + 1$. Let $x \in \mathbb{P}$. Then, for each $n \geq 1$,*

$$M_t(x)_n = n^t \inf\Big\{\sum_{k=1}^n x_k a_k^p : 0 < a \in \mathbb{R}^n, \sum_{k=1}^n a_k = 1\Big\}.$$

Proof. Suppose $0 < a \in \mathbb{R}^n$, and $\sum_{k=1}^n a_k = 1$. Let $q = p/(p-1) = p/t$. Then, by Hölder's inequality,

$$\begin{aligned} 1 &= \sum_{j=1}^n (a_j^p x_j)^{1/p} x_j^{-1/p} \\ &\leq \Big(\sum_{j=1}^n a_j^p x_j\Big)^{1/p} \Big(\sum_{j=1}^n x_j^{-q/p}\Big)^{1/q}, \end{aligned}$$

Hence

$$\frac{1}{\left(\sum_{j=1}^n x_j^{-q/p}\right)^{p/q}} \leq \sum_{j=1}^n a_j^p x_j.$$

Equality holds here if

$$a_j = \frac{1}{x_j^{q/p}\left(\sum_{j=1}^n x_j^{-q/p}\right)}, \; j = 1, 2, \ldots, n.$$

It follows that

$$\begin{aligned} \frac{1}{\left(\sum_{j=1}^n \frac{1}{x_j^{1/t}}\right)^t} &= \frac{1}{\left(\sum_{j=1}^n x_j^{-q/p}\right)^{p/q}} \\ &= \inf\Big\{\sum_{k=1}^n x_k a_k^p : 0 < a \in \mathbb{R}^n, \sum_{k=1}^n a_k = 1\Big\}. \end{aligned}$$

The stated result follows. □

An equivalent formulation is that, with $p = t + 1$,

$$M_t(x)_n = n^{p-1} \inf\Big\{\sum_{k=1}^n x_k a_k^p : 0 < a \in \mathbb{R}^n, \sum_{k=1}^n a_k = 1\Big\}. \tag{1}$$

Thus

$$M_t(x)_n \leq n^{p-1} \sum_{k=1}^n x_k a_k^p \tag{2}$$

for all probability vectors $a \in \mathbb{R}^n$.

Remark. Already this result reveals that M_t is super-additive and hence concave.

The result we want to prove is the following: if $x \in \mathbb{P}$,

$$\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{\sum_{j=1}^{k}\frac{1}{x_j^{1/t}}}\right)^t = \frac{1}{n}\sum_{k=1}^{n} M_t(x)_k \le M_t(A(x))_n$$
$$= \left(\frac{n}{\sum_{k=1}^{n}\frac{1}{A(x)_k^{1/t}}}\right)^t,\ n = 1, 2, \ldots,$$

with equality if and only if x is a constant sequence.

Our idea is this: with n fixed, suppose a is a probability vector in $\mathbb{R}^n$. Then, by the previous lemma, with $p = t + 1$,

$$\begin{aligned} M_t(A(x))_n &\le n^{p-1}\sum_{j=1}^{n} a_j^p A(x)_j \\ &= n^{p-1}\sum_{j=1}^{n}\frac{a_j^p}{j}\sum_{k=1}^{j} x_k \\ &= n^{p-1}\sum_{k=1}^{n} x_k \sum_{j=k}^{n}\frac{a_j^p}{j}, \end{aligned}$$

after interchanging the order of summation, and there is equality here for a suitable a. But, also, if $u_i \in \mathbb{R}^i$ is a probability vector,

$$M_t(x)_i \le i^{p-1}\sum_{j=1}^{i} u_{ij}^p x_j,\ i = 1, 2, \ldots, n,$$

whence

$$\begin{aligned} \sum_{i=1}^{n} M_t(x)_i &\le \sum_{i=1}^{n} i^{p-1}\sum_{j=1}^{i} u_{ij}^p x_j \\ &= \sum_{j=1}^{n} x_j \sum_{i=j}^{n} i^{p-1} u_{ij}^p. \end{aligned}$$

So, we can accomplish our objective if, given a probability vector $a \in \mathbb{R}^n$, we can construct similar vectors $u_i \in \mathbb{R}^i$ so that

$$\sum_{i=j}^{n} i^{p-1} u_{ij}^p \le n^p \sum_{k=j}^{n}\frac{a_k^p}{k},\ j = 1, 2, \ldots, n. \tag{3}$$

To reach our goal, and to show that these inequalities can be solved, we construct a certain lower triangular row-stochastic matrix from a probability vector. To this end, we use the following result due to Kedlaya [8]:

Lemma 2. *The rational numbers*

$$\alpha_k(i,j) = \frac{\binom{n-i}{j-k}\binom{i-1}{k-1}}{\binom{n-1}{j-1}},\ 1 \le i,j,k \le n,$$

satisfy the following conditions

(1) $\alpha_k(i,j) \ge 0$, *for all* i,j,k;

(2) $\alpha_k(i,j) = 0$ *for all* $k > \min(i,j)$;

(3) $\alpha_k(i,j) = \alpha_k(j,i)$ *for all* i,j,k;

(4) $\sum_{k=1}^{n} \alpha_k(i,j) = 1$ *for all* i,j;

(5)
$$\sum_{i=1}^{n} \alpha_k(i,j) = \begin{cases} \frac{n}{j}, & \textit{for } 1 \le k \le j, \\ 0, & \textit{for} k > j. \end{cases}$$

Given a probability vector $a \in \mathbb{R}^2$, construct the $n \times n$ matrix $A = [a_{ij}]$ by

$$a_{ij} = \sum_{k=1}^{n} \alpha_j(i,k) a_k,\ 1 \le i,j \le n.$$

Then each row of A is a probability vector, because

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} &= \sum_{j=1}^{n}\sum_{k=1}^{n} \alpha_j(i,k) a_k \\ &= \sum_{k=1}^{n} a_k \sum_{j=1}^{n} \alpha_j(i,k) \\ &= \sum_{k=1}^{n} a_k \text{ (by 4.)} \\ &= 1 \end{aligned}$$

for all i. Also, $a_{ij} = 0$ for all $j > i$. Thus A is a lower triangular row-stochastic matrix.

But, for each pair of indices i,j, a_{ij} is a convex combination of $a_1, a_2, \ldots, a_n$, and so, if $p \ge 1$,

$$a_{ij}^p \le \sum_{k=1}^{n} \alpha_j(i,k) a_k^p,\ i,j = 1,2,\ldots,n.$$

Hence

$$\begin{aligned}
\sum_{i=j}^{n} i^{p-1} a_{ij}^{p} &= \sum_{i=1}^{n} i^{p-1} a_{ij}^{p} \\
&\leq n^{p-1} \sum_{i=1}^{n} \sum_{k=1}^{n} \alpha_j(i,k) a_k^p \\
&= n^{p-1} \sum_{k=1}^{n} a_k^p \sum_{i=1}^{n} \alpha_j(i,k) \\
&= n^p \sum_{k=j}^{n} \frac{a_k^p}{k} \text{ (by 5.).}
\end{aligned}$$

Looking back at (3) we now see that we can solve this by selecting

$$u_{ij} = a_{ij}, \ j = 1, 2, \ldots, i.$$

We're now ready to provide a proof of the theorem.

Fix $x \in \mathbb{P}$, and a positive integer n. Let a be a probability vector in $\mathbb{R}^n$. Choose the corresponding lower triangular row-stochastic matrix $A = [a_{ij}]$ as above. By Lemma 1, if $1 \leq i \leq n$,

$$M_t(x)_i \leq i^{p-1} \sum_{j=1}^{i} a_{ij}^p x_j.$$

Hence

$$\begin{aligned}
\sum_{i=1}^{n} M_t(x)_i &\leq \sum_{i=1}^{n} i^{p-1} \sum_{j=1}^{i} a_{ij}^p x_j \\
&= \sum_{j=1}^{n} x_j \sum_{i=j}^{n} i^{p-1} a_{ij}^p \\
&\leq \sum_{j=1}^{n} x_j n^p \sum_{k=j}^{n} \frac{a_k^p}{k} \\
&= n^p \sum_{k=1}^{n} a_k^p \frac{1}{k} \sum_{j=1}^{k} x_j \\
&= n^p \sum_{k=1}^{n} a_k^p A(x)_k.
\end{aligned}$$

Whence

$$n^{-t-1}\sum_{i=1}^{n} M_t(x)_i$$

is a lower bound for the set

$$\Big\{\sum_{k=1}^{n} a_k^p A(x)_k : 0 < a \in \mathbb{R}^n,\ \sum_{k=1}^{n} a_k = 1\Big\},$$

whose infimum is $n^{-t}M_t(A(x))_n$. Hence

$$\frac{1}{n}\sum_{i=1}^{n} M_t(x)_i \le M_t(A(x))_n,$$

and we're done, apart from dealing with the case of equality, which is easily settled.

4. A Number of Corollaries

We deduce a number of special cases of Theorem 1.

Corollary 1.

$$A \circ \min \le \min \circ A.$$

This is obtained by letting $t \to 0^+$.

Corollary 2 (Kedlaya)**.**

$$A \circ G \le G \circ A.$$

i.e., $\forall x \in \mathbb{P}$,

$$\frac{1}{n}\sum_{i=1}^{n} G(x)_i \le G(A(x))_n,\ \forall x \in \mathbb{P},\ \forall n \ge 1;$$

or, more explicitly,

$$\frac{1}{n}\sum_{i=1}^{n} \sqrt[i]{\prod_{j=1}^{i} x_j} \le \sqrt[n]{\prod_{j=1}^{n} \frac{1}{j}\sum_{i=1}^{j} x_i}.$$

This is obtained by letting $t \to \infty$. This implies Carleman's classical inequality [3, 5, 7]:

$$||G(x)||_1 \le e||x||_1,\ \forall x \in \mathbb{P}.$$

Corollary 3 (Mond & Pečarić)**.**

$$A \circ H \le H \circ A.$$

This is obtained by letting $t = 1$. It says that, $\forall x \in \mathbb{P}$,

$$\frac{1}{n}\sum_{k=1}^{n} \frac{k}{\sum_{j=1}^{k} \frac{1}{x_j}} \leq \frac{n}{\sum_{k=1}^{n} \frac{k}{\sum_{j=1}^{k} x_j}},\ n = 1, 2, \ldots$$

Since the sequence

$$\sum_{j=1}^{k} x_j,\ k = 1, 2, \ldots$$

is strictly increasing we deduce that the right-hand side does not exceed

$$\frac{n\sum_{j=1}^{n} x_j}{\sum_{k=1}^{n} k} = \frac{2\sum_{j=1}^{n} x_j}{n+1},$$

whence

$$\sum_{k=1}^{n} \frac{k}{\sum_{j=1}^{k} \frac{1}{x_j}} \leq \frac{2n}{n+1}\sum_{j=1}^{n} x_j < 2\sum_{j=1}^{n} x_j,$$

which gives a solution to Zinn's Monthly problem. Moreover, the constant 2 cannot be replaced by a smaller number, as can be seen by taking $x_j = 1/j$, $j = 1, 2, \ldots, n$. Thus, if $x \in \mathbb{P} \cap \ell_1$, so does $H(x)$, and $||H(x)||_1 < 2||x||_1$.

Corollary 4. *$\forall t > 0$ and $\forall x \in \mathbb{P}$,*

$$\sum_{i=1}^{n} M_t(x)_i \leq \left(\frac{n^{1+1/t}}{\sum_{j=1}^{n} j^{1/t}}\right)^t \sum_{k=1}^{n} x_k,\ n = 1, 2, \ldots$$

and the inequality is strict unless $n = 1$.

Proof.

$$\begin{aligned}
nM_t(A)_n &= n^p \left(\frac{1}{\sum_{j=1}^{n} A_j^{-1/t}}\right)^t \\
&= n^p \left(\frac{1}{\sum_{j=1}^{n} \frac{j^{1/t}}{(\sum_{k=1}^{j} x_k)^{1/t}}}\right)^t \\
&\leq n^p \sum_{k=1}^{n} x_k \left(\frac{1}{\sum_{j=1}^{n} j^{1/t}}\right)^t \\
&= \left(\frac{n^{1+1/t}}{\sum_{j=1}^{n} j^{1/t}}\right)^t \sum_{k=1}^{n} x_k.
\end{aligned}$$

$\square$

Since

$$\lim_{n\to\infty} \frac{n^{1+1/t}}{\sum_{j=1}^{n} j^{1/t}} = 1 + \frac{1}{t},$$

a simple consequence of the fact that, with $s = 1/t$,

$$\frac{\sum_{j=1}^{n} j^s}{n^{s+1}} = \frac{1}{n}\sum_{j=1}^{n}(\frac{j}{n})^s$$

is a Riemann sum for the integral

$$\int_0^1 x^s\, dx = \frac{1}{1+s},$$

Corollary 4 implies a result of Knopp [10] to the effect that

$$||M_t(x)||_1 \leq (1 + 1/t)^t ||x||_1, \ \forall x \in \mathbb{P}. \tag{4}$$

5. Companion Results When $t < 0$

The means M_t also make sense when $t < 0$. Similar methods to those employed in the previous section lead to the following statement.

Theorem 2. *If $-1 < t < 0$, then $A \circ M_t \geq M_t \circ A$. Moreover, $A \circ M_t(x) = M_t \circ A(x)$ if and only if $x = \lambda e$ for some $\lambda > 0$.*

Letting $p = -1/t$, we can recast this in terms of p: If $p \geq 1$, then, for all $x \in \mathbb{P}$, and all $n \geq 1$,

$$\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{i=1}^{k} x_i\right)^p\right)^{1/p} \leq \frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{i=1}^{k} x_i^p\right)^{1/p}. \tag{5}$$

There is equality only when x is a constant sequence. This is a substantial improvement of a very well-known result due to Hardy [4, 5], which states that, if $x \in \ell_p$, then $A(x) \in \ell_p$, and

$$||A(x)||_p \leq \frac{p}{p-1}||x||_p.$$

Inequality (5) was found by Bennett [2], who pointed out that the reversed inequality holds when $0 < p < 1$. A stronger form of (5) was established by B. Mond and J. E. Pečarić [11], and a weighted version of their result was outlined by Kedlaya [9]. But results of this kind were announced much earlier by Nanjundiah [13], though he appears not to have published a proof.

REFERENCES

[1] G. BENNETT, Factorizing the Classical Inequalities, *Mem. Amer. Math. Soc.* **120** (1996), *no. 576*, viii+130pp.

[2] G. BENNETT, Summability matrices and random walk, *Houston J. Math* **28** (2002), *no. 4*, 865–898.

[3] T. CARLEMAN, Sur les functions quasi-analytiques, Fifth Scand. Math. Congress (1923), 181–196.

[4] G. H. HARDY, Notes on some points of the integral calculus (LX), *Messenger of Math.* **54** (1925), 150–156.

[5] G. H. HARDY, J. E. LITTLEWOOD, AND G. PÓLYA, *Inequalities*, Cambridge University Press, 1934.

[6] F. HOLLAND, On a mixed arithmetic-mean, geometric-mean inequality, *Math. Competitions* **5** (1992), 60–64.

[7] M. JOHANSSON, L. E. PERSSON, A. WEDESTIG, Carleman's inequality—history, proofs and some new generalizations, *J. Ineq. Pure & Appl. Math.* **4** (3), (2003), 1–19.

[8] K. KEDLAYA, Proof of a mixed arithmetic-mean, geometric-mean inequality, *Amer. Math. Monthly* **101** (1994), 355–357.

[9] K. KEDLAYA, A weighted mixed-mean inequality, *Amer. Math. Monthly* **106** (1999), 355–358.

[10] K. KNOPP, Über Reihen mit positiven Gliedern, *J. London Math. Soc.*, **3** (1928), 205-211.

[11] B. MOND AND J. E. PEČARIĆ, A mixed means inequality, *Austral.Math. Soc. Gaz.*, **23**(2) (1996), 67–70.

[12] B. MOND AND J. E. PEČARIĆ, A mixed arithmetic-mean-harmonic-mean inequality, *Lin. Algebra Appl.* **237/238** (1996), 449–454.

[13] T. S. NANJUNDIAH, Sharpening of some classical inequalities, *Math. Student* **20** (1952), 24–25.

Finbarr Holland,
Mathematics Department,
University College,
Cork, Ireland
f.holland@ucc.ie

Received on 8 June 2006.