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Lattice Polygons in the Plane and the Number 12

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1. Introduction

A convex polygon $\mathcal{P}$ in $\mathbb{R}^2$ all of whose vertices have integer coordinates is called a convex lattice polygon. If the polygon has n lattice points on its boundary, represented by the vectors $p_1, \dots, p_n$ (in anticlockwise order), then we say that the length $l(\mathcal{P})$ is n. The dual (convex) lattice polygon $\mathcal{P}^\vee$ is by definition the convex hull of the difference vectors $q_i = p_{i+1} - p_i$, where indices throughout the note are considered modulo n. In this note we will give a simple proof of the fact that if $\mathcal{P}$ be a convex lattice polygon in $\mathbb{R}^2$ whose only interior lattice point is the origin, then $l(\mathcal{P}) + l(\mathcal{P}^\vee) = 12$. This result has its origin in a correspondence between convex lattice polygons and certain toric varieties, and has several ingenious proofs (see [1] and [2]). These proofs use either Noether's formula or modular forms to explain the occurrence of the number 12. Our proof observes that if $l(\mathcal{P})$ increases by one then $l(\mathcal{P}^\vee)$ decreases by one, so that their sum is constant. The number 12 then appears as 3 (the smallest possible length) plus 9 (the largest possible length.) Our proof (which grew out of the second authors project, supervised by the first author) also supports the suggestion in [2] that the number 3 can be viewed as a discrete analogue of π in this context.

2. The Theorem

Theorem 2.1. *Let $\mathcal{P}$ be a convex lattice polygon in $\mathbb{R}^2$ whose only interior lattice point is the origin, and let $\mathcal{P}^\vee$ be its dual. Then $l(\mathcal{P}) + l(\mathcal{P}^\vee) = 12$.*

It is not hard to see that up to the action of $GL(2, \mathbb{Z})$ there are 16 such polygons but we will not need this fact. We will however use the following result of P. Scott [3], the proof of which relies only on elementary geometry.

Lemma 2.1. *Let $\mathcal{P}$ be a convex lattice polygon in the plane with $l(\mathcal{P})$ boundary lattice points and $c \geq 1$ interior points then $l(\mathcal{P}) \leq 2c + 7$.*

We will also use picks formula, which in our context states:

Lemma 2.2. *The area of a lattice polygon $\mathcal{P}$ is given by the formula $A(\mathcal{P}) = c + l(\mathcal{P})/2 - 1$.*

In our case $c = 1$, so that $l(\mathcal{P}) \leq 9$ and $A(\mathcal{P}) = l(\mathcal{P})/2$. Since we will think of the plane as the complex plane $\mathbb{C}$, we will label the vectors p_i and q_i as z_i and w_i respectively and we let γ_i denote the straight line path (parameterised on $[0,1]$) joining z_i to z_{i+1}. We will fix $z_1 = 1$ throughout unless we state otherwise. We recall that for a piecewise smooth curve γ in $\mathbb{C}$ we have $\int_\gamma \bar{z}\, dz = 2iA$ where A denotes the area enclosed by γ. In particular a convex lattice polygon with vertices $z_1, \ldots, z_n$ has area

$$-\frac{i}{2}\sum_{i=1}^{n}\int_0^1 \overline{(tz_{i+1} + (1-t)z_i)}(z_{i+1} - z_i)dt$$

$$= -\frac{i}{4}\sum_{i=1}^{n}[|z_{i+1}|^2 - |z_i|^2 + \bar{z}_i z_{i+1} - z_i \bar{z}_{i+1}]$$

$$= -\frac{i}{4}\sum_{i=1}^{n}[\bar{z}_i z_{i+1} - z_i \bar{z}_{i+1}].$$

It will be convenient to introduce the notation $A_{ij} = -\frac{i}{4}(\bar{z}_i z_j - z_i \bar{z}_j)$, namely the signed area of the oriented triangle with vertices o, z_i, z_j. Similarly, we let $A^\vee_{ij} = -\frac{i}{4}(\bar{w}_i w_j - w_i \bar{w}_j)$, so that

$$A^\vee_{ii+1} = -\frac{i}{4}(\bar{z}_i z_{i+1} - z_i \bar{z}_{i+1} + \bar{z}_{i+1} z_{i+} - z_{i+1}\bar{z}_{i+2} + \bar{z}_{i+2} z_i - z_i \bar{z}_{i+2}).$$

In summary $A^\vee_{ii+1} = A_{ii+2} + A_{i+1i+2} + A_{i+2i}$, so that $l(\mathcal{P}^\vee) = 2l(\mathcal{P}) - \sum_{i=1}^n A_{ii+2}$. This immediately yields that when $l(\mathcal{P}) = n = 3$ we have $l(\mathcal{P}^\vee) = A^\vee_{12} + A^\vee_{23} + A^\vee_{31} = (A_{12} + A_{23} + A_{31}) + (A_{23} + A_{31} + A_{12}) + (A_{31} + A_{12} + A_{23}) = 3(A_{31} + A_{12} + A_{23}) = 3 \times 3 = 9$ (since $A_{ii+1} = 1$ for all i). If $l(\mathcal{P}) = n = 4$, $\sum_{i=1}^n A_{ii+2} = A_{13} + A_{24} + A_{31} + A_{42} = 0$ (since $A_{ij} = -A_{ji}\ \forall i,j$) and we have $l(\mathcal{P}^\vee) = 2l(\mathcal{P}) = 8$. In future we will denote the sum $\sum_{i=1}^n A_{ii+2}$ by $d(\mathcal{P})$, and show that it increases by 3 when $l(\mathcal{P})$ increases by 1, thus keeping $l(\mathcal{P}) + l(\mathcal{P}^\vee)$ constant.

Proof. The case $l(\mathcal{P}) = n = 5$ is the crucial case as all others essentially follow from this one. Here since $A(\mathcal{P}) = 5/2$ and no edge contains more than 3 vertices, we may centre $\mathcal{P}$ at the origin with its vertices on the unit square. Since one of the 4 lines $x = 0$, $y = 0$, $y = \pm x$ must contain 2 of the 5 vertices of $\mathcal{P}$, we will assume that it is the x-axis. We may also assume that 2 (z_2 and z_3) of the remaining 3 vertices lie above the x-axis and that the remaining vertex z_5 lies below it. It is immediate that $A_{41} = 0$ (z_4, o, and z_1 being collinear). We first consider the case where $\mathcal{P}$ has an edge of length 2 (i.e., containing 3 vertices). This edge consists either of the points $\{z_5, z_1, z_2\}$ or $\{z_3, z_4, z_5\}$, so that either $A_{52} = 2$ or $A_{35} = 2$. In the former case $A_{35} = -1$, and in the latter $A_{52} = -1$. In both cases the remaining A_{ii+2} are equal to 1, so that $\sum_{i=1}^{n} A_{ii+2} = 3$ and we are done. When $\mathcal{P}$ has no edge of length 2, we must have an additional $A_{ii+2} = 0$ on account of the fact that z_5 doesn't lie on a vertical edge of length 2. This forces the remaining A_{ii+2} to be 1, and again $\sum_{i=1}^{n} A_{ii+2} = 3$. When $l(\mathcal{P}) = 6$, we can (after relabelling if necessary) delete the vertex z_6 from $\mathcal{P}$ to obtain a convex lattice polygon $\mathcal{P}'$ containing the origin, and $d(\mathcal{P}) - d(\mathcal{P}') = A_{13} + A_{35} + A_{51} + A_{24} + A_{46} + A_{62} - (A_{13} + A_{35} + A_{52} + A_{24} + A_{41}) = A_{51} + A_{46} + A_{62} + A_{25} + A_{14} = 3$. The last equality follows from the observation that $A_{51} + A_{46} + A_{62} + A_{25} + A_{14} = d(\mathcal{P}'')$ where $\mathcal{P}''$ is either the convex lattice pentagon $\{z_1, z_2, z_4, z_5, z_6\}$ containing the origin, so that $d(\mathcal{P}'') = 3$ by above, or else the convex lattice hexagon (the above pentagon with the origin adjoined) with no interior lattice point, where the computation is trivial. We now have $l(\mathcal{P}^\vee) = 2l(\mathcal{P}) - (d(\mathcal{P}') + 3) = 12 - 6 = 6$. It is intriguing that the cases $l(\mathcal{P}) = 7, 8, 9$ are identical to that of $l(\mathcal{P}) = 6$. In each case, just as above $d(\mathcal{P}) - d(\mathcal{P}') = d(\mathcal{P}'')$ where $\mathcal{P}''$ is either a convex lattice pentagon containing the origin, or the convex lattice hexagon with no interior lattice point, so that $d(\mathcal{P}) - d(\mathcal{P}') = 3$. $\square$

Remark. Finally we point out some connections with [2]. There it is shown that the vectors p_i and p_{i+1} form a basis for the lattice $\mathbb{Z}^2$ with the same orientation as the standard basis $\{(1,0),(0,1)\}$, and that there are matrices $M_i = \begin{pmatrix} 0 & 1 \\ -1 & d_i \end{pmatrix}$ in $SL(2,\mathbb{Z})$ such that $M_i \begin{pmatrix} p_{i-1} \\ p_i \end{pmatrix} = \begin{pmatrix} p_i \\ p_{i+1} \end{pmatrix}$, where the 2×2 matrices $\begin{pmatrix} p_i \\ p_{i+1} \end{pmatrix}$ have the row vectors p_i and p_{i+1} as their rows. In addition we have that

$$\begin{pmatrix} 0 & 1 \\ -1 & d_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & d_{n-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & d_1 \end{pmatrix} = I,$$

the identity matrix. It is a simple matter to see that in fact $d_i = A_{ii+2}$ and that the above equations therefore contain some of our observations above. When $n = 3$ they imply that $A_{13} = A_{32} = A_{21} = -1$. When $n = 4$ they imply that $A_{ii+2} = -A_{i+2i}\ \forall i$. When $n = 5$ they imply that the multi-set $\{A_{13}, A_{35}, A_{52}, A_{24}, A_{41}\}$ is either the multi-set $\{-1, 2, 1, 0, 1\}$ or else $\{0, 1, 1, 0, 1\}$. In either case $d(\mathcal{P}) = \sum_{i=1}^{n} A_{ii+2} = 3$. We also observe that our proof supports the suggestion in [2] that 3 can be viewed as a discrete analogue of π. Clearly $d(\mathcal{P})$ has an interpretation as the sum of the "discrete exterior angles" of $\mathcal{P}$ (as defined in [2].) We have shown that increasing $l(\mathcal{P})$ by one increases this sum by 3, whereas the sum of the exterior angles increases by π for a general polygon.

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