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Old, Recent and New Results on Quasinormal Subgroups

STEWART E. STONEHEWER

1. Survey of Published Results

A subgroup A of a group G is *quasinormal* (or *permutable*) in G if

$$AX = XA$$

for all subgroups X of G, or equivalently for all *cyclic* subgroups X of G. This is also equivalent to requiring that the subgroup $\langle A, X\rangle$ generated by A and X is simply the product AX. In this situation we write

$$A \; qn \; G.$$

Thus normal subgroups are always quasinormal, but not conversely. For, if p is a prime, then any cyclic group C_{p^n} extended by any cyclic group C_{p^m} has all subgroups quasinormal (provided, when $p = 2$ and $n \geqslant 2$, the cyclic subgroup of order 4 in C_{2^n} is central in the extension). The same is true if C_{p^n} is replaced by any abelian p-group H of finite exponent, with C_{p^m} acting on H as a group of power automorphisms (and elements of order 4 in H are again central in the extension if $p = 2$). These results can be found in sections 2.3 and 2.4 of [16].

One of the earliest results about quasinormal subgroups is due to Ore, who proved in 1938 that *a quasinormal subgroup of a finite group G is always subnormal in G* ([14]). When G is infinite, then A does not have to be subnormal, but it is always ascendant in G (see [17]). Clearly the extent to which a quasinormal subgroup A can differ from being normal is of interest and a measure of this was given by Itô and Szép in 1962 when they proved that, again with G finite, and denoting the *core* of A in G by A_G,

the quotient A/A_G is always nilpotent

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(see [11]). This was significantly improved in 1973 when Maier and Schmid proved a stronger result, viz.

A/A_G lies in the hypercentre of G

(see [13]). When G is not finite, research is still ongoing (see [12], [16] and particularly [4]).

It was John Thompson ([22]) who first exhibited (finite) examples with A/A_G not abelian. In fact his examples had nilpotency class 2 and they were the starting point for much further research. Examples with A/A_G of nilpotency class c, for any positive integer c, were constructed in [3] and [18]; and examples with A/A_G of derived length d, for any $d \geqslant 1$, were constructed in [19]. All these were finite p-groups. Then Berger and Gross ([1]) generalised the construction in [19] to give a universal embedding theorem:

Let p be a prime and let X be a cyclic group of order p^n. Then there exists a finite p-group $G_1 = A_1X_1$, where A_1 qn G_1, $(A_1)_{G_1} = 1$, $X_1 \cong X$; and any finite group

$$G = AX, \tag{1}$$

with A qn G and $A_G = 1$, embeds in G_1, with A embedding in A_1 and X mapping onto X_1.

Here G_1, A_1 (and of course X_1) depend only on p and n. But clearly, the form (1) is very restricted. It has to do only with the structure of subgroups generated by A and one other element, and gives little information about how a quasinormal subgroup A embeds in an arbitrary finite group G. Thus in 1998 Busetto and Napolitani began to study the normal closure A^G of a quasinormal subgroup A of a finite group G, starting with the case A cyclic. When p is prime and G is a finite p-group, they conjectured that the commutator subgroup

$[A, G]$ is abelian.

Three years later, Cossey and Stonehewer proved that, with $A = \langle a\rangle$,

$$[A, G] = \{[a, g] \mid g \in G\}, \tag{2}$$

provided p is odd. Thus let $g \in G$ and consider $H = A\langle g\rangle$, a product of two cyclic subgroups and so metacyclic by [10]. Therefore

$$\langle [a, g]\rangle = H' \lhd H$$

and it follows that A normalises every cyclic subgroup, and hence every subgroup of $[A, G]$. But then A^G $(= A[A, G])$ normalises every subgroup of $[A,G]$ and so $[A,G]$ is a Dedekind group, i.e. all of its subgroups are normal. Therefore $[A,G]$ is abelian, since p is odd (see [16], Theorem 2.3.12). Thus the following is proved in [6].

Theorem 1. *If A is a cyclic quasinormal subgroup of odd order in a finite group G, then $[A,G]$ is abelian and A acts on it by conjugation as a group of power automorphisms.*

It is interesting to observe that difficulties in proving the above result only arise when

$$A \cap [A,G] = 1.$$

For,

let A be any quasinormal subgroup of any group G,

and suppose that $A \cap [A,G] = 1$. Let X be any subgroup of $[A,G]$. Then

$$[A,X] \leqslant AX \cap [A,G] = X.$$

So A normalises every subgroup of $[A,G]$ and therefore so also does A^G. Hence $[A,G]$ is a Dedekind group. We claim that

$$[A,G] \textit{ is abelian.}$$

For, if not, then $[A,G]$ is the direct product of the quaternion group of order 8, an elementary abelian 2-group and an abelian group with all its elements of odd order (see [15], 5.3.7). Certainly A centralises the elements of order 2 in $[A,G]$. Let X be a cyclic subgroup of order 4 in $[A,G]$. If A doesn't centralise X, then $A/C_A(X)$ has order 2 and $AX/C_A(X)$ is the dihedral group of order 8 with $A/C_A(X)$ a non-central subgroup of order 2. But this contradicts the quasinormality of A. Therefore A must centralise X and so A centralises the 2-component of $[A,G]$. Hence A^G does the same, forcing $[A,G]$ to be abelian.

Theorem 1 can fail when A has even order.

Example ([7]). *There is a group G of order 2^{17} with a cyclic quasinormal subgroup A of order 2^7 and $[A,G]' = A^{64} = \Omega(A)$.*

Thus $[A,G]$ here has nilpotency class 2 and just fails to be abelian. In fact, given a very natural hypothesis (which this example satisfies), this is as bad as it gets, as the following result from [7] shows.

Theorem 2. *If A is a cyclic quasinormal subgroup of a finite group G, then provided the condition* (3) *below holds, $[A,G]'$ has order at* *most 2 and lies in A; and A acts on $[A,G]/[A,G]'$ as a group of power automorphisms.*

Suppose that $X = \langle x\rangle$ is a cyclic 2-subgroup of G and let $A_2 = \langle a_2\rangle$ be the 2-component of A. Then, like *all* subgroups of A, A_2 is quasinormal in G ([16], Lemma 5.2.11) and $H = A_2X$ is metacyclic ([16], Theorem 5.2.13). Thus $H' = \langle [a_2,x]\rangle$. We must exclude the dihedral action of X on H' here. So we require

$$[H',X] \leqslant (H')^4. \tag{3}$$

Without (3), we still get a lot of information. If we denote by $A_{2'}$ the $2'$-component of A, then the following is proved in [8].

Theorem 3. *Let A be a cyclic quasinormal subgroup of a finite group G. Then*

(i) $[A,G] = [A_2,G]\times[A_{2'},G]$;

(ii) $[A_{2'},G]$ *is an abelian $2'$-group;*

(iii) *A acts on $[A_{2'},G]$ as a group of power automorphisms;*

(iv) $[A_2,G]$ *is a 2-group of class at most 2;*

(v) $[A_2,G,A]$ *is normal in G and lies in the centre of $[A_2,G]$; and*

(vi) *A acts on $[A_2,G,A]$ as a group of power automorphisms.*

Remark. It was shown by Cooper in [5] that a power automorphism of a finite abelian group is always *universal*, i.e. each element maps to the *same* power. Therefore all the power automorphisms in Theorems 1, 2 and 3 are universal.

There appears to be no known example, satisfying the hypotheses of Theorem 3, in which $[A,G]'$ has order greater than 2. In other words, it is an open question whether condition (3) is necessary in the hypotheses of Theorem 2. With further research it should be possible to answer this question.

2. Recent and Ongoing Developments

Before moving on to an investigation of non-cyclic quasinormal subgroups, it is natural to consider the importance of the finiteness of G in the theorems of the previous section. Certainly one would expect some difference between the rôles played by elements of finite and elements of infinite order. In many ways, the presence of the latter is a bonus. For example, in any group G with a quasinormal subgroup A, an infinite cyclic subgroup of G which intersects A trivially actually normalises A ([17]). It turns out that allowing G to be arbitrary, results separate naturally into the two cases where the cyclic subgroup A is finite or infinite. In the former situation, it is shown in [21] that the hypothesis that G is finite can be omitted from each of Theorems 1, 2 and 3.

Theorem 4. *Theorems* 1, 2 *and* 3 *in Section* 1 *do not require G to be finite, only that A be finite. The statements in all cases remain unchanged.*

When G is finite, induction arguments on order play an important part. When G is infinite we proceed as follows. To generalise Theorems 1, 2 and 3 to get Theorem 4, we may always assume that G is finitely generated. But then $| A^G : A |$ is always finite, by [12], Lemma 7.1.9. So A^G is finite, since A is finite. This result enables induction arguments to proceed.

Another key result in proving Theorem 4 extends a theorem by Busetto [2]. Let A be a cyclic quasinormal subgroup of order p^n in a group G. Busetto shows that if $A_G = 1$, then $A^G \leqslant Z_{2n}(G)$, the $2n$-th term of the upper central series of G. In fact we can say more. For, if $| A_G | = p^m$ with $m < n$, then

$$A^G \leqslant Z_{2n-m}(G)$$

(see [21]).

When A is an infinite cyclic quasinormal subgroup, then the results are even better. The following will appear in [21].

Theorem 5. *Let A be an infinite cyclic quasinormal subgroup of a group G. Then $[A, G]$ is abelian and A acts on it as a group of power automorphisms. Also $[A, G]$ is periodic if and only if $A \cap [A, G] = 1$; and if $[A, G]$ is not periodic (i.e., $A \cap [A, G] \neq 1$), then A^G is abelian.*

The power automorphisms in Theorem 5 are not always universal. For example let H be an abelian group of type p^∞ and let a be a p-adic integer such that $a \equiv 1 \bmod p$ ($a \equiv 1 \bmod 4$ if $p = 2$). Thus

$$h \mapsto h^a, \tag{4}$$

for all $h \in H$, defines an automorphism of H. Let G be the split extension of H by the cyclic group A generated by a. Then every subgroup of G is quasinormal (see [16], Theorem 2.4.11). Clearly the power automorphism (4) is not universal in general.

On the other hand, all the power automorphisms in the statement of Theorem 4 are universal. For, $[A, G]$ has finite exponent; and a power automorphism of an abelian p-group of finite exponent is universal (see [15], 13.4.3 (ii)). Let H, K be groups of finite exponent r, s respectively, with $(r,s)=1$. Let $h \mapsto h^m$, all $h \in H$, be a universal power automorphism of H; and let $k \mapsto k^n$, all $k \in K$, be a universal power automorphism of K. Then there are integers i and j such that

$$m-n=ir+js.$$

Put $t=m-ir$. Thus $h^t=h^m$ and $k^t=k^n$ and combining the automorphisms of H and K gives a *universal* power automorphism of $H \times K$.

If A is a cyclic quasinormal subgroup of any group G, then, as stated above, all subgroups of A are quasinormal in G. But which subgroups of an arbitrary quasinormal subgroup are also quasinormal? If A is abelian, there is the following (see [20]).

Theorem 6. *Let A be an abelian quasinormal subgroup of any group G (finite or infinite) and let n be a positive integer, either odd or divisible by 4. Then A^n is also quasinormal in G.*

There are examples (see [20]) which show that this result can fail when $n \equiv 2 \bmod 4$. When A is a finite abelian p-group, then $A^p = \Phi(A)$, the Frattini subgroup of A. So $\Phi(A)$ is quasinormal in G if p is odd. One might conjecture that it is unnecessary for A to be abelian here. However, there are finite p-groups, with p odd, with a quasinormal subgroup A of class 2 such that $\Phi(A)$ is *not* quasinormal ([9]). But in these examples there is a subgroup B of index p in A which is quasinormal in G. Thus we are led to ask the following questions.

Let A be a quasinormal subgroup of a finite p-group G and suppose that A is not normal in G. Let

$$A_G = B_0 < B_1 < \cdots < B_n = A \tag{5}$$

be an unrefinable chain of quasinormal subgroups of G. What can be said about the subgroups B_i? Certainly $B_i \lhd B_{i+1}$ for all i. But do different chains have the same length? And if so, are they isomorphic (as in the Jordan–Hölder Theorem)?

These questions seem to be difficult to answer. It is natural to start with the case where G is a finite p-group (and this is probably the hardest case). Also having A abelian, even elementary of rank 2, seems a sensible starting point. Thus let G be a finite p-group and let A be a quasinormal subgroup of G, with A elementary abelian of rank 2. Is there a quasinormal subgroup of G of order p lying in A? Suppose that p is odd and let G be a group of minimal order for which the answer is negative. Then $A_G = 1$. Also it is possible to show that

G has a unique minimal normal subgroup N.

Let $a \in A$, $a \neq 1$. Thus, bearing in mind that a quasinormal subgroup of order p normalises every subgroup of G, there is a cyclic subgroup $X = \langle x \rangle$ of G such that $X^a \neq X$. Therefore $A \cap X = 1$. But there is an element $b \in A$ such that $X^b = X$. Hence $A = \langle a \rangle \times \langle b \rangle$. Similarly there is a cyclic subgroup $Y = \langle y \rangle$ of G such that $Y^b \neq Y$; and there is an element $c = a^i b^j$ $(\neq 1)$ such that $Y^c = Y$. Clearly $p \nmid i$ and so $A = \langle b, c \rangle$. Also $X^c \neq X$. Therefore we may assume that

$$A = \langle a \rangle \times \langle b \rangle,$$

where $Y^a = Y$, $X^a \neq X$, $X^b = X$, $Y^b \neq Y$.

If $\langle A, x, y \rangle < G$, then by choice of G there is an element $d \in A$ of order p which normalises X and Y. Since $\langle d \rangle \neq \langle a \rangle$, we have $A = \langle a, d \rangle$ and so A normalises Y, a contradiction. Therefore

$$G = \langle A, x, y \rangle.$$

In fact it is possible to show that $A \leqslant \Phi(G)$ and hence

$$G = \langle x, y \rangle.$$

Note that $A \cap X = A \cap Y = 1$. Suppose that

$$N \cap AX = N \cap AY = 1.$$

Again by choice of G there is a subgroup D of order p in A such that DN/N is quasinormal in G/N. Therefore DNX is a subgroup and so also is

$$DNX \cap AX = DX.$$

Similarly DY is a subgroup. Clearly $D \neq \langle b \rangle$ and so $A = \langle b, D \rangle$. But then A normalises X, again a contradiction.

Thus without loss of generality we may assume that

$$N \leqslant AX.$$

By considering the various remaining possibilities for exactly where N lies, we can make similar progress. There are five cases of increasing complexity; and for the final case we have currently to restrict to the situation where G' is elementary abelian. (This facilitates the use of elementary modular represtentation theory.) We have obtained the following in [9].

Theorem 7. *Let G be a finite p-group, with p an odd prime, and with G' elementary abelian. Let A be a quasinormal subgroup of G, with A elementary abelian of rank 2. Then A contains a quasinormal subgroup of G of order p.*

Thinking of this statement as saying that A contains a subgroup of index p which is quasinormal in G, then the programme from here is clear. First extend Theorem 7 to metabelian p-groups G, then to p-groups G of arbitrary class, then to A elementary abelian of arbitrary finite rank, then to A abelian, then to A arbitrary, then to $p = 2$. If this can be done, then $\mid B_{i+1} : B_i \mid= p$ in (5) and the questions following (5) have affirmative answers for p-groups G. The next step would be to move to arbitrary finite groups G and then possibly to infinite groups. It will be a long programme.

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Stewart E. Stonehewer,
University of Warwick
Coventry CV4 7AL,
England