Groups with Few Normalizer Subgroups

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Dedicated to Martin L. Newell

Abstract. The behaviour of normalizer subgroups of a group has often a strong influence on the structure of the group itself. In this paper groups with finitely many normalizers of subgroups with a given property χ are investigated, for various relevant choices of the property χ.

1. Introduction

A subgroup X of a group G is called *almost normal* if it has finitely many conjugates in G, or equivalently if its normalizer $N_G(X)$ has finite index in G. In a famous paper of 1955, B. H. Neumann [17] proved that all subgroups of a group G are almost normal if and only if the centre $Z(G)$ has finite index, and the same conclusion holds if the restriction is imposed only to abelian subgroups (see [10]). Thus central-by-finite groups are precisely those groups in which all the normalizers of (abelian) subgroups have finite index, and this result suggests that the behaviour of normalizers has a strong influence on the structure of the group. In fact, Y.D. Polovickiĭ [18] has shown that if a group G has finitely many normalizers of abelian subgroups, then the factor group $G/Z(G)$ is finite.

Recently, groups have been considered with finitely many normalizers of subgroups with a given property (see [5], [6], [7], [8], [9]); the aim of this paper is to give a survey exposition of the main theorems on this subject. In particular, in Section 2 we consider groups in which all but finitely many normalizers of abelian subgroups have finite index, obtaining as a corollary the theorem of Polovickiĭ quoted

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above, while in Section 3 we describe groups with finitely many normalizers of non-abelian subgroups, with special emphasis on the case of groups in which every subgroup is either normal or abelian. The last two sections are devoted to the structure of groups with finitely many normalizers of subnormal or non-subnormal subgroups, respectively.

Most of our notation is standard and can for instance be found in [20].

2. Normalizers of Abelian Subgroups

The following result plays a central role in the study of groups with finitely many normalizers of subgroups with a given property. It was proved by B. H. Neumann [16] in the more general case of groups covered by finitely many cosets, and can for instance be used to prove that every group covered by finitely many abelian subgroups is central-by-finite.

Lemma 2.1. *Let the group $G = X_1 \cup \ldots \cup X_t$ be the union of finitely many subgroups $X_1, \ldots, X_t$. Then any X_i of infinite index can be omitted from this decomposition; in particular, at least one of the subgroups $X_1, \ldots, X_t$ has finite index in G.*

Our first main result is taken from [6] and describes groups containing only finitely many normalizers of abelian subgroups which are not almost normal. Recall that the *FC-centre* of a group G is the subgroup consisting of all elements of G having only finitely many conjugates, and a group G is called an *FC-group* if G coincides with its FC-centre. Thus a group G has the property FC if and only if the centralizer $C_G(x)$ has finite index in G for each element x; in particular, abelian-by-finite FC-groups are central-by-finite.

Theorem 2.2. *Let G be a group in which all but finitely many normalizers of abelian subgroups have finite index. Then the factor group $G/Z(G)$ is finite.*

Proof. Let $N_G(X_1), \ldots, N_G(X_k)$ be the normalizers of infinite index of abelian subgroups of G, and let F be the FC-centre of G. Clearly,

$$G = F \cup N_G(X_1) \cup \ldots \cup N_G(X_k)$$

and so it follows from Lemma 2.1 that $G = F$ is an FC-group. Another application of Lemma 2.1 yields that the set $N_G(X_1) \cup \ldots \cup$

$N_G(X_k)$ is properly contained in G. Let x be an element of

$$G \setminus (N_G(X_1) \cup \ldots \cup N_G(X_k)),$$

and consider any infinite abelian subgroup A of the centralizer $C_G(x)$; as x normalizes A, the normalizer $N_G(A)$ must have finite index in G. Thus all abelian subgroups of $C_G(x)$ are almost normal, and so $C_G(x)$ is central-by-finite (see [10]). On the other hand, the index $|G : C_G(x)|$ is finite, so that G is an abelian-by-finite FC-group and hence $G/Z(G)$ is finite. □

Polovickiĭ's theorem is obviously a consequence of Theorem 2.2.

Corollary 2.3. *Let G be a group with finitely many normalizers of abelian subgroups. Then the factor group $G/Z(G)$ is finite.*

If G is an arbitrary group, the *norm* $N(G)$ of G is the intersection of all the normalizers of subgroups of G; this concept was introduced by R. Baer, and it is well known that $Z(G) \leq N(G) \leq Z_2(G)$. It follows from Corollary 2.3 that if G is a group such that $G/N(G)$ is finite, then also $G/Z(G)$ is finite (of course this is also a direct consequence of Neumann's theorem quoted in the introduction). Some evidence of the fact that the factor group $N(G)/Z(G)$ is usually small is given by a recent result of J. C. Beidleman, H. Heineken and M. L. Newell (see [1], Theorem 2).

3. Normalizers of Non-abelian Subgroups

This section is devoted to the study of groups with finitely many normalizers of non-abelian subgroups. The first step is of course the consideration of groups in which every non-abelian subgroup is normal. Groups with this property are called *metahamiltonian groups*; they were introduced and investigated by G. M. Romalis and N. F. Sesekin ([22], [23], [24]), who proved that every locally soluble metahamiltonian group is soluble with derived length at most 3 and its commutator subgroup is finite with prime-power order. As the original papers of Romalis and Sesekin are in Russian language and not easily available, we give here a proof of their main result in the case of locally graded groups. Recall that a group G is *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index; thus all locally (soluble-by-finite) groups are locally graded. Clearly, Tarski groups (i.e., infinite simple groups whose proper non-trivial subgroups have prime order) are metahamiltonian, and the assumption that the group is locally graded is necessary in order to avoid this and other similar pathologies.

Lemma 3.1. *Let G be a residually finite metahamiltonian group. Then G is either nilpotent or abelian-by-finite, and in particular it locally satisfies the maximal condition on subgroups.*

Proof. Suppose that G is not abelian-by-finite, and let $\mathfrak{H}$ be the set of all subgroups of finite index of G; then each element H of $\mathfrak{H}$ is normal in G and G/H is a Dedekind group. Therefore

$$\gamma_3(G) \leq \bigcap_{H \in \mathfrak{H}} H = \{1\}$$

and hence G is nilpotent. □

Lemma 3.2. *Let G be a metahamiltonian group, and let A be a finitely generated torsion-free abelian normal subgroup of G. Then A is contained in $Z(G)$.*

Proof. Assume by contradiction that there exists an element x of G such that $[A, x] \neq \{1\}$, and suppose first that $A \cap \langle x \rangle = \{1\}$. Clearly, there exists an odd prime number p such that $[A^{p^n}, x] \neq \{1\}$ for all positive integers n; then for each n the subgroup $A^{p^n}\langle x \rangle$ is normal in G and $G/A^{p^n}\langle x \rangle$ is a Dedekind group. As $p > 2$, it follows that $[A, x]$ is contained in $A^{p^n}\langle x \rangle$ for all n and so also in $\langle x \rangle$. Therefore $[A, x] = \{1\}$, and this contradiction shows that

$$A \cap \langle x \rangle = \langle x^m \rangle \neq \{1\}.$$

Let

$$A/A \cap \langle x \rangle = E/A \cap \langle x \rangle \times B/A \cap \langle x \rangle,$$

where $E/A \cap \langle x \rangle$ is finite and $B/A \cap \langle x \rangle$ is torsion-free. As $A \cap \langle x \rangle$ is contained in $Z(\langle x, A \rangle)$ and $\langle E, x \rangle / A \cap \langle x \rangle$ is finite, by Schur's theorem we have that $[E, x]$ is a finite subgroup of A (see [20], Part 1, Theorem 4.12), and so $[E, x] = \{1\}$. On the other hand, A/E is a torsion-free abelian normal subgroup of $\langle x, A \rangle / E$ and $\langle xE \rangle \cap A/E = \{1\}$, so that it follows from the first part of the proof that $[A, x] \leq E$. Therefore $[A, x, x] = \{1\}$, and hence

$$[A, x]^m = [A, x^m] = \{1\}.$$

Thus $[A, x] = \{1\}$ and this last contradiction completes the proof of the lemma. □

Lemma 3.3. *Let G be a metahamiltonian group with finite commutator subgroup. Then the order of G' is a prime-power number.*

Proof. As G' is finite, there exists a finitely generated subgroup E of G such that $E' = G'$. Moreover, $E/Z(E)$ is finite and $Z(E)$ contains a torsion-free subgroup A of finite index; clearly $G' = E' \simeq E'A/A$ and hence replacing G by E/A, it can be assumed without loss of generality that G is finite. If X is any Sylow p-subgroup of G, by hypothesis either X is normal in G or $N_G(X) = C_G(X)$ and in the latter case G is p-nilpotent (see for instance [21], 10.1.8). It follows that G contains a normal non-trivial Sylow subgroup P, and by the Schur-Zassenhaus Theorem there exists a subgroup Q of G such that $G = Q \ltimes P$. If Q is abelian, G' is contained in P and hence it has prime-power order. Suppose that Q is not abelian, so that $G = P \times Q$. If P is abelian, G' is contained on Q and by induction on the order of G we have that G' has prime-power order. Assume finally that also P is not abelian. Then G/P and G/Q are Dedekind groups and hence G' has order at most 4. The lemma is proved. □

Theorem 3.4. *Let G be a locally graded metahamiltonian group. Then G is soluble with derived length at most 3 and the commutator subgroup G' of G is finite with prime-power order.*

Proof. Suppose first that G is soluble. In order to prove that G' is finite, it can be assumed by induction on the derived length of G that G'' is finite; replacing G by G/G'' we may also suppose that G is metabelian. Let E be a finitely generated non-abelian subgroup of G. Then E is normal in G and G/E is a Dedekind group; as E is residually finite (see [20], Part 2, Theorem 9.51), it follows from Lemma 3.1 that E is polycyclic and hence G' is finitely generated. Thus G itself can be assumed to be finitely generated and so even polycyclic. Moreover, Lemma 3.1 yields that G contains a torsion-free nilpotent normal subgroup N of finite index. Let A be a maximal abelian normal subgroup of N, so that $C_N(A) = A$; on the other hand, A is contained in $Z(N)$ by Lemma 3.2 and hence $N = A$ is abelian. It follows again from Lemma 3.2 that N lies in $Z(G)$, so that $G/Z(G)$ is finite and hence G' is finite.

In the general case, let $\mathfrak{X}$ be the set of all non-abelian subgroups of G, and put

$$M = \bigcap_{X \in \mathfrak{X}} X.$$

Clearly, each element X of $\mathfrak{X}$ is a normal subgroup of G and G/X is a Dedekind group, so that M is normal in G and $G'' \leq M$. Moreover, M is a locally graded group whose proper subgroups are abelian, so that M is either abelian or finite. Therefore G is soluble-by-finite. Let R be the largest soluble normal subgroup of G. If R is a subgroup of $Z(G)$, it follows that $G/Z(G)$ is finite, so that G' is finite and hence G is soluble by Lemma 3.3. Suppose that R is not contained in $Z(G)$, and let x be an element of G such that $[R,x] \neq \{1\}$; the soluble subgroup $R\langle x\rangle$ is not abelian, so that it is normal in G and in particular $R\langle x\rangle = R$. Thus R is not abelian and G/R is a Dedekind group, so that G is soluble also in this case. It follows now from the first part of the proof and from Lemma 3.3 that G' is a finite subgroup with prime-power order.

Assume finally for a contradiction that $G^{(3)} \neq \{1\}$ and let a and b be elements of G'' such that $[a,b] \neq 1$. Then $\langle a,b\rangle$ is normal in G and $G/\langle a,b\rangle$ is a Dedekind group, so that $G'' = \langle a,b\rangle$ and G'/G'' has order 2. As G' is nilpotent, it follows that $G'' = \{1\}$ and this contradiction completes the proof of the theorem. □

Of course, nothing can be said about the solubility of groups with finitely many normalizers of non-abelian subgroups; moreover, the derived length of soluble groups with this property cannot be bounded. Therefore the following result proved in [5] is the best possible extension of Theorem 3.4 to this class of groups.

Theorem 3.5. *Let G be a locally graded group with finitely many normalizers of non-abelian subgroups. Then the commutator subgroup G' of G is finite.*

The consideration of the infinite dihedral group shows that a group with finite conjugacy classes of non-abelian subgroups may have infinite commutator subgroup. However, if we denote by $N^*(G)$ the *non-abelian norm* of the group G (i.e. the intersection of all the normalizers of non-abelian subgroups of G), the above theorem has the following consequence, that can also be considered as a generalization of the famous Schur's theorem on the finiteness of the derived subgroup of central-by-finite groups.

Corollary 3.6. *Let G be a locally graded group such that the factor group $G/N^*(G)$ is finite. Then the commutator subgroup G' of G is finite.*

It must be mentioned here that B. Bruno and R. E. Phillips [2] extended the investigation of Romalis and Sesekin, considering groups whose non-normal subgroups are locally nilpotent. They worked within the universe of $\mathcal{W}$-groups (where $\mathcal{W}$ is the class of all groups in which every finitely generated non-nilpotent subgroup has a finite non-nilpotent homomorphic image), and proved that in this case groups with the above property either are locally nilpotent or have finite commutator subgroup. Note that all locally (soluble-by-finite) groups have the property $\mathcal{W}$. In this context, the following result on normalizer subgroups has been obtained in [9].

Theorem 3.7. *Let G be a $\mathcal{W}$-group with finitely many normalizers of non-(locally nilpotent) subgroups. Then either G is locally nilpotent or its commutator subgroup G' is finite.*

It follows that if a $\mathcal{W}$-group G has finitely many normalizers of non-(locally nilpotent) subgroups, then G even has only finitely many normalizers of non-nilpotent subgroups. However, it seems difficult to deal with locally nilpotent groups with finitely many normalizers of non-nilpotent subgroups; in fact, there exist (soluble) groups with trivial centre in which all proper subgroups are nilpotent and subnormal (see [13]) and such groups can have arbitrarily high derived length (see [14]).

4. Normalizers of Subnormal Subgroups

A group G is called a *T-group* if normality is a transitive relation in G, i.e., if all subnormal subgroups of G are normal. The structure of soluble T-groups has been described by W. Gaschütz [12] in the finite case and by D. J. S. Robinson [19] for arbitrary groups. It turns out in particular that soluble groups with the property T are metabelian and hypercyclic, and that finitely generated soluble T-groups either are finite or abelian. In recent years, many authors have investigated the structure of soluble groups in which normality is imposed only on certain relevant systems of subnormal subgroups; other classes of generalized T-groups can be introduced by imposing that the set of all subnormal non-normal subgroups of the group is small in some sense.

The *Wielandt subgroup* $\omega(G)$ of a group G is defined to be the intersection of all the normalizers of subnormal subgroups of G; this concept is naturally analogous to the norm of a group. Clearly, $\omega(G)$ is a T-group, and G is a T-group if and only if it coincides with its Wielandt subgroup; thus the size of $G/\omega(G)$ can be considered as a measure of the distance of the group G from the property T. For instance, it is well known that if G is a group satisfying the minimal condition on subnormal subgroups, then $G/\omega(G)$ is finite (see [20], Part 1, Theorem 5.49). Moreover, if G is a group such that $G/\omega(G)$ is finite, it is clear that each subnormal subgroup of G has only finitely many conjugates; conversely, C. Casolo proved that if a soluble group G has boundedly finite conjugacy classes of subnormal subgroups, then the Wielandt subgroup $\omega(G)$ has finite index in G (see [3], Theorem 4.8). Of course, the finiteness of $G/\omega(G)$ also implies that the group G has finitely many normalizers of subnormal subgroups, and groups with this latter property generalize those in which normality is a transitive relation. The following result has recently been proved in [8]; it shows in particular that for a periodic soluble group G the finiteness of the set of normalizers of subnormal subgroups is equivalent to that of the index $|G:\omega(G)|$.

Theorem 4.1. *Let G be a soluble group with finitely many normalizers of subnormal subgroups. If G locally satisfies the maximal condition on subgroups, then the Wielandt subgroup $\omega(G)$ has finite index in G.*

A better result can be proved for finitely generated soluble groups with finitely many normalizers of subnormal subgroups. In fact, it turns out that such groups are central-by-finite, and so in particular finitely generated torsion-free soluble groups with finitely many normalizers of subnormal subgroups must be abelian (see [8]).

5. Normalizers of Non-subnormal Subgroups

It is well known that a finite group is nilpotent if and only if all its subgroups are subnormal but, as we mentioned at the end of Section 3, there exist infinite groups with trivial centre and all subgroups subnormal. The structure of groups in which every subgroup is subnormal has recently obtained much attention; among many other deep results, W. Möhres [15] proved that such groups are soluble.

As a contribution to the theory of groups with few non-subnormal subgroups, in this section we will describe groups with finitely many normalizers of non-subnormal subgroups; it turns out that such groups coincide with those in which every subgroup either is subnormal or has finitely many conjugates (see [7]), and groups with this latter property have been completely characterized in [11].

Recall that the *Baer radical* of a group G is the subgroup generated by all abelian subnormal subgroups of G, and G is called a *Baer group* if it coincides with its Baer radical. In particular, every Baer group is locally nilpotent and all its cyclic subgroups are subnormal. The following lemma shows in particular that groups with finitely many normalizers of non-subnormal subgroups either are Baer groups or have the property FC.

Lemma 5.1. *Let G be a group in which all but finitely many normalizers of non-subnormal subgroups have finite index. Then G is either a Baer group or an FC-group.*

Proof. Let B and F be the Baer radical and the FC-centre of G, respectively. If x is any element of the set $G \setminus (B \cup F)$, the subgroup $\langle x \rangle$ is not subnormal and the index $|G : N_G(\langle x \rangle)|$ is infinite. Therefore

$$G = B \cup F \cup N_G(X_1) \cup \ldots \cup N_G(X_k),$$

where $N_G(X_1), \ldots, N_G(X_k)$ are all normalizers of infinite index of non-subnormal subgroups of G. It follows from Lemma 2.1 that $G = B \cup F$, so that either $G = B$ is a Baer group or $G = F$ is an FC-group. □

Theorem 5.2. *For a group G the following statements are equivalent:*

(i) *G has finitely many normalizers of non-subnormal subgroups.*
(ii) *All but finitely many normalizers of non-subnormal subgroups of G have finite index.*
(iii) *Every subgroup of G is either subnormal or almost normal.*

The above theorem allows us to give a complete description of groups with finitely many normalizers of non-subnormal subgroups. In fact, it follows from Proposition 2.2 and Theorem 2.8 of [11] that a group G has this property if and only if satisfies one of the following conditions:

(a) all subgroups of G are subnormal;
(b) the factor group $G/Z(G)$ is finite;
(c) G is periodic and contains a nilpotent normal subgroup N of finite index and class 2 whose commutator subgroup is cyclic with prime-power order p^k; moreover, the Fitting subgroup F of G has index a power of p and $N' \leq \langle x \rangle$ for each element x of $G \setminus F$.

Since a torsion-free group in which all subgroups are subnormal is nilpotent (see [4] or [25]), we have the following consequence.

Corollary 5.3. *Let G be a torsion-free group in which all but finitely many normalizers of non-subnormal subgroups have finite index. Then G is nilpotent.*

If G is any group, we shall denote by $\omega^*(G)$ the intersection of all the normalizers of non-subnormal subgroups of G (with the stipulation that $\omega^*(G) = G$ if all subgroups of G are subnormal).

Corollary 5.4. *Let G be a group with finitely many normalizers of non-subnormal subgroups. Then the factor group $G/\omega^*(G)$ is finite.*

Proof. Clearly, it can be assumed that G contains subgroups which are not subnormal, so that it is an FC-group by Lemma 5.1 and Theorem 5.2. Then by the above description we may also suppose G contains a nilpotent normal subgroup N of finite index such that N' lies in every non-subnormal subgroup of G. As G/N' is central-by-finite, it follows that $G/\omega^*(G)$ is finite. □

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