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The $F^{a,b,c}$ Conjecture, I

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Abstract. In 1977 a five-part conjecture was made about a family of groups related to trivalent graphs and one part of the conjecture was proved. The conjecture completely determines all finite members of the family. Here we prove another part of the conjecture and foreshadow a paper which completes the proof of the other three parts.

1. Introduction

R M Foster became interested in symmetrical graphs which could be used as electrical networks in the 1920s, and worked on this subject for many years. At a conference held in Waterloo, Ontario, in April 1966, Foster presented a census of symmetric trivalent graphs with up to 400 vertices. H S M Coxeter, who had one of the few copies of Foster's census, became interested. As part of Coxeter's investigation he defined the groups $F^{a,b,c}$ by

$$F^{a,b,c} = \langle r, s \mid r^2, rs^a rs^b rs^c \rangle.$$

These arose because some of the groups have Cayley diagrams which are '0-symmetric' or 'faithful'. Campbell, Coxeter and Robertson investigated the groups $F^{a,b,c}$ in [1] and, after determining the structure of various subclasses, made 'the $F^{a,b,c}$ conjecture' which we state after some preliminaries. Combined with some results in [1] this conjecture completely describes the structure of all finite groups in the $F^{a,b,c}$ family in terms of a specific finite quotient which is fully understood.

Define $n = a + b + c$ and $d = (a - b, b - c)$. The structure of the groups

$$H^{a,b,c} = \langle r, s \mid r^2, s^{2n}, rs^a rs^b rs^c \rangle$$

is completely determined in Section 3 of [1]. If $n = 0$ then $F^{a,b,c}$ is clearly infinite. In [1] the structure of $F^{a,b,c}$ is fully described for the situation where the greatest common divisor of a, b and c is not one, including all finite cases. The conjecture addresses the remaining cases. Provided $(a,b,c) = 1$, $n \neq 0$ and $(d,6) \neq 6$, the groups $H^{a,b,c}$ are finite metabelian groups. If $d \geq 6$ the groups $F^{a,b,c}$ are infinite [1].

The $F^{a,b,c}$ conjecture is as follows. Suppose $(a,b,c) = 1$ and $n \neq 0$. Let

$$\theta : F^{a,b,c} \to H^{a,b,c}$$

be the natural homomorphism. Let $N = \ker\theta$. Then

$$\begin{aligned} &N = 1 \text{ if } d = 1, \\ &N = 1 \text{ if } d = 2, \\ &N \cong C_2 \text{ if } d = 3, \\ &N \cong Q_8 \text{ if } d = 4, \\ &N \cong SL(2,5) \text{ if } d = 5. \end{aligned}$$

The conjecture was proved true when $d = 1$ in [3], see also Corollary 3.4 of [4] for an alternative proof. Many special cases supporting the conjecture have been proved, see [1], [4], [5], [9], and [10]. Here we present a proof that the conjecture holds when $d = 5$. The proof was suggested from a study of small cases investigated with the ACE coset enumerator [6], as available in GAP [7]. The way ACE helped was by indicating that the following steps were true in specific instances. The proof of the conjecture in the cases $d = 2, 3$ and 4 is quite different in nature from that presented in this paper since in these cases $(d,6) \neq 1$. The proof for these remaining three cases will appear in [8], where we make use of computer-generated proofs for specific instances to motivate our general proofs.

2. Proof of the Conjecture when $d = 5$

In what follows we assume that $d = 5$ and $N = \ker\theta$. First we indicate the strategy behind our proof by breaking the proof into a number of steps.

Step 1. s^{2n} commutes with rs^5r.
Step 2. s^{10n} is central in $F^{a,b,c}$.
Step 3. $u = s^{2n}$, $v = rs^{2n}r$ generate N.
Step 4. $u^5, v^5, (uv)^3$ and $(vu^2)^2$ are central in N.
Step 5. Put $M = \langle u^5, v^5, (uv)^3, (vu^2)^2\rangle$. Then $N/M \cong A_5$.
Step 6. N is perfect.

STEP 7. M is the multiplier of A_5.
STEP 8. $N \cong SL(2,5)$.

We proceed to prove each of these steps in turn. We will use the notation $x \sim y$ to mean that x commutes with y.

Proof of Step 1. From $r^2 = 1$ and $rs^a rs^b rs^c = 1$ we have

$$(s^a rs^b)(s^{-c} rs^{-b}) = (rs^{-c}r)(rs^a r) = rs^{a-c}r.$$

Hence $s^a(rs^{b-c}r)s^{-b} = rs^{a-c}r$.

Similarly $s^b(rs^{c-a}r)s^{-c} = rs^{b-a}r$ and $s^c(rs^{a-b}r)s^{-a} = rs^{c-b}r$. From the first and third of these we have $s^{2a}(rs^{b-a}r)s^{-b-c} = rs^{a-c}r$, and using the second of the three relations

$$s^{2a+b}(rs^{c-a}r)s^{-b-2c} = rs^{a-c}r.$$

Hence

$$s^{2a+b}s^{b+2c}(rs^{a-c}r)s^{-2a-b}s^{-b-2c} = rs^{a-c}r$$

showing that $s^{2n} \sim rs^{a-c}r$. Similarly $s^{2n} \sim rs^{b-a}r$ so, since $5 = d = (a-c, b-a)$, $s^{2n} \sim rs^5 r$.

This step already appears as Lemma 1.1 of [5] based on details which are in the proof of Theorem 4.1 of [1].

Proof of Step 2 (Theorem 2.6 of [5]). Since $a \equiv b \equiv c \pmod 5$ we have $n \equiv 3a \pmod 5$ so $n \equiv 0 \pmod 5$ if and only if $5 \mid (a,b,c)$ showing that n is coprime to 5. From Step 1, $s^{2n} \sim rs^5 r$ so $s^5 \sim rs^{2n}r$. Hence $s^{10n} \sim rs^5 r$ and $s^{10n} \sim rs^{2n}r$. Since $(5, 2n) = 1$, we have $s^{10n} \sim rsr$. We now see that $s^{10n} \sim rs^a r$, so $s^{10n} \sim s^{-c}rs^{-b}$, showing that $s^{10n} \sim r$ and so s^{10n} is central in $F^{a,b,c}$.

Proof of Step 3. To prove that $\langle u, v\rangle = N$ we need to show that $N = \langle u\rangle^F = \langle u, v\rangle$. Put $\widehat{N} = \langle u, v\rangle$. Then $rur = v \in \widehat{N}$ and $s^{-1}us = u \in \widehat{N}$. Also $rvr = u \in \widehat{N}$. It remains to consider $s^{-1}vs$.

Now $s^5 \sim rs^{2n}r$ so, if $a \equiv b \equiv c \equiv 1 \pmod 5$, $s^{2n-1} \sim rs^{2n}r$. Hence

$$s^{-1}rs^{2n}rs = s^{-2n}.rs^{2n}r.s^{2n} = u^{-1}vu.$$

Similarly if $a \equiv b \equiv c \equiv 2 \pmod 5$, $s^{-1}rs^{2n}rs = u^{-3}vu^3$, while $a \equiv b \equiv c \equiv 3 \pmod 5$ gives $s^{-1}rs^{2n}rs = u^{-2}vu^2$ and $a \equiv b \equiv c \equiv 4 \pmod 5$ gives $s^{-1}rs^{2n}rs = u^{-4}vu^4$.

Proof of Step 4. First we need a lemma.

Lemma 2.1. (i) *s^{2n} commutes with $rs^{2n}rs^{2n}r$.*
(ii) *s^{2n} commutes with $rs^{2n}rs^{4n}rs^{2n}r$.*

Proof. (i) s^{2n} commutes with $rs^a rs^b r$ since it commutes with s^c. Since $a \equiv b \equiv c \pmod 5$ we see that

$$2n - a \equiv a + 2b + 2c \equiv 0 \pmod 5$$

so $s^{2n} \sim rs^{2n-a}r$.

Similarly $s^{2n} \sim rs^{2n-b}r$ so

$$s^{2n} \sim rs^{2n-a}r.rs^a rs^b r.rs^{2n-b}r = rs^{2n}rs^{2n}r.$$

(ii) $rs^{2n}rs^{4n}rs^{2n}r = (rs^{2n}rs^{2n}r)^2$. □

We need to check that $u^5, v^5, (uv)^3$ and $(vu^2)^2$ are central in N. The first two are easy since $u^5 = s^{10n}$ which is central in N by Step 2. Also $v^5 = rs^{10n}r$. But s^{10n} is central in $F^{a,b,c}$ by Step 2 so $v^5 = s^{10n}$ which is central in N.

Before proving the final two elements are central we prove another lemma.

Lemma 2.2. $(s^{2n}r)^6 = (rs^{2n}rs^{4n})^2$.

Proof.

$$\begin{aligned}(s^{2n}r)^6 &= s^{2n}.rs^{2n}rs^{2n}r.s^{2n}.rs^{2n}rs^{2n}r\\ &= rs^{2n}rs^{2n}r.rs^{2n}rs^{2n}r.s^{4n}\\ &= rs^{2n}rs^{4n}rs^{2n}rs^{4n}\\ &= (rs^{2n}rs^{4n})^2\end{aligned}$$

□

By Lemma 2.2 we see that $(uv)^3 = (vu^2)^2$ so to prove these elements central it suffices to examine one of them.

$$\begin{aligned}u(vu^2)^2 &= s^{2n}rs^{2n}rs^{4n}rs^{2n}rs^{4n}\\ &= rs^{2n}rs^{4n}rs^{2n}rs^{6n}\\ &= (vu^2)^2u\end{aligned}$$

Also

$$\begin{aligned}v(vu^2)^2 &= rs^{2n}r.rs^{2n}rs^{4n}rs^{2n}rs^{4n}\\ &= rs^{4n}rs^{4n}rs^{2n}rs^{4n}\end{aligned}$$

But

$$\begin{aligned}(vu^2)^2v &= rs^{2n}rs^{4n}rs^{2n}rs^{4n}.rs^{2n}r\\ &= rs^{4n}rs^{4n}rs^{2n}rs^{4n}.\end{aligned}$$

Hence $v(vu^2)^2 = (vu^2)^2v$ as required.

Proof of Step 5. Certainly M is a normal subgroup of N since its generators are central. Thus N/M is a homomorphic image of

$$L \cong \langle u, v \mid u^5, v^5, (uv)^3, (vu^2)^2 \rangle.$$

It is easy to see, by coset enumeration, that $L \cong A_5$. It remains to prove that N is nontrivial. Suppose, by way of contradiction, that N is trivial. Then, in this case $F^{a,b,c} \cong H^{a,b,c}$, so $F^{a,b,c}$ is metabelian. However adding the relation $s^5 = 1$ to the presentation for $F^{a,b,c}$ gives A_5 as an image of the metabelian group $F^{a,b,c}$ which is the necessary contradiction.

Proof of Step 6. To prove that N is perfect we add the relation $u \sim v$ to the relations for N to give N' and prove that $u = v = 1$ in N'.

Now $u \sim v$ gives $s^{2n} \sim rs^{2n}r$. But we also have $s^{2n} \sim rs^5r$ by Step 1 so, since $(2n, 5) = 1$, we have $s^{2n} \sim rsr$.

However $r = s^a rs^b rs^c$ and s^{2n} commutes with the right hand side so $s^{2n} \sim r$ proving that s^{2n} is central.

Also $rs^{2n}r = s^{2n}$ so $u = v$. That $s^{2n} = 1$ now follows from Theorem 3.3 of [4] in which the Schur multiplier of $H^{a,b,c}$ is shown to be trivial.

This proves that N is perfect.

Proof of Step 7. We have $M \leq Z(N)$ and $M \leq N'$. Also $N/M \cong A_5$, so M is contained in the multiplier of A_5. M is nontrivial because otherwise $F^{a,b,c}$ is an extension of A_5 by $H^{a,b,c}$ which has multiplier C_2 (see [2]). This contradicts the fact that $F^{a,b,c}$ has deficiency zero.

Proof of Step 8. This follows from what has been proved above.

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References

[1] C. M. Campbell, H. S. M. Coxeter, E. F. Robertson, Some families of finite groups having two generators and two relations, *Proc. Roy. Soc. London Ser. A* **357** (1977), 423–438.

[2] C. M. Campbell, E. F. Robertson, Classes of groups related to $F^{a,b,c}$, *Proc. Roy. Soc. Edinburgh Sect. A* **78** (1977/78), 209–218.

[3] C. M. Campbell, E. F. Robertson, On 2-generator 2-relation soluble groups, *Proc. Edinburgh Math. Soc. (2)* **23** (1980), 269–273.

[4] C. M. Campbell, E. F. Robertson, Groups related to $F^{a,b,c}$ involving Fibonacci numbers, *The geometric vein*, Springer, New York-Berlin (1981), 569–576.

[5] C. M. Campbell, E. F. Robertson, On the $F^{a,b,c}$ conjecture, *Mitt. Math. Sem. Giessen* **164** (1984), 25–36.

[6] Greg Gamble, Alexander Hulpke, George Havas, Colin Ramsay, *GAP package ACE; Advanced Coset Enumerator*, 2004; `http://www.gap-system.org/Packages/ace.html`

[7] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version* 4.4, 2004; `http://www.gap-system.org/`

[8] George Havas, Edmund F. Robertson, Dale C. Sutherland, The $F^{a,b,c}$ conjecture, II, *in preparation.*

[9] Manley Perkel, Groups of type $F^{a,b,-c}$, *Israel J. Math.* **52** (1985), 167–176.

[10] Fulvia Spaggiari, On certain classes of finite groups, *Ricerche Mat.* **46** (1997), 31–43.

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