A Characterisation of Injectors of Finite Soluble Groups

REX DARK

We aim to describe the result indicated in the title, which was obtained in collaboration with Arnold Feldman. Most of the article is devoted to a description of part of the theory of injectors, with references to the book by Doerk and Hawkes [2], which gives a comprehensive exposition of the topic (though they arrange the proofs in an order different from ours).

All the groups considered will be assumed finite, and after this paragraph they will also be soluble. One of the most fundamental and useful results about finite groups G is Sylow's theorem, which given a prime number p, guarantees the existence and conjugacy of subgroups P (called *Sylow p-subgroups*) such that the order $|P|$ is a power of p but the index $|G : P|$ is not divisible by p. It is therefore interesting to investigate situations in which this result can, or cannot, be generalised. For example, instead of a single prime number p, consider a set π of prime numbers, and recall that P is a *π-group* if all the prime factors of $|P|$ are in π. Then a *Hall π-subgroup* of G is a π-subgroup P such that $|G : P|$ is not divisible by any prime number in π. We also recall that a finite group G is *soluble* if every non-trivial quotient group G/K has a non-trivial abelian normal subgroup A/K. It was proved by P. Hall [2, I (3.3)] that Hall π-subgroups exist and are conjugate in every finite soluble group. He also showed [2, I (3.6)] that a finite group G is soluble if and only if it has Hall π-subgroups for all sets of primes π, so giving a remarkable connection between solubility (involving abelian factor groups A/K) and the existence of subgroups with given orders.

The theory of Hall subgroups can be used to illuminate the structure of soluble groups, and it was generalised by W. Gaschütz [2, III §3] to prove the existence and conjugacy of *covering subgroups* and *projectors*. We shall not define these concepts, but we note that they helped to inspire the 'dual' theory of Fischer subgroups and injectors, which is discussed below. However we shall briefly describe the striking special case of Carter subgroups, which were discovered before the general theory was developed. We recall that a finite group G is *nilpotent* if every non-trivial quotient group G/K has a non-trivial centre Z/K. It was proved by W. Burnside [2, A (8.3)] that every finite nilpotent group is the direct product of its Sylow subgroups, which means that for each prime number p, the Sylow p-subgroup is normal and unique. As an example of Hall's theorem, we note that a nilpotent group has a unique Hall π-subgroup, got by taking the direct product of the Sylow p-subgroups with $p \in \pi$. Moreover a finite group is nilpotent if and only if all its Sylow subgroups are normal, so giving another remarkable connection between nilpotency (involving central factor groups Z/K) and the existence of normal subgroups with given orders. Finally we recall that a subgroup $H \leq G$ is *self-normalising* if there is no subgroup $N \neq H$ such that $H \lhd N \leq G$, and we remark that a nilpotent group has no proper self-normalising subgroups.

Definition. A *Carter subgroup* is a subgroup which is nilpotent and self-normalising (and must therefore be a maximal nilpotent subgroup, because it cannot be contained in a bigger nilpotent subgroup, by the last remark).

Theorem. (CARTER [2, III (4.6)]) *Carter subgroups exist and are conjugate in every finite soluble group.*

If P is a Hall π-subgroup of G, and N is another π-subgroup such that P normalises N, then PN is a subgroup of G, and its order is

$$|PN| = |PN/N| \cdot |N| = |P/P \cap N| \cdot |N|,$$

which has all its prime factors in π; thus PN is a π-subgroup. But the Hall subgroup P is a maximal π-subgroup, and therefore $N \leq P$. This proves a Hall π-subgroup contains every π-subgroup which it normalises, and it can be shown that this property characterises Hall π-subgroups. The following definition and theorem are suggested by this characterisation.

Definition. A *nilpotent Fischer subgroup* of G is a nilpotent subgroup H which contains every nilpotent subgroup of G which it normalises (and must therefore be a maximal nilpotent subgroup).

Theorem. (FISCHER [2, VIII (4.8)]) *Nilpotent Fischer subgroups exist and are conjugate in every finite soluble group.*

Thus in every finite soluble group, the Carter subgroups and the nilpotent Fischer subgroups form conjugacy classes of maximal nilpotent subgroups. For example, in the symmetric group S_3, the Sylow 2-subgroups are the Carter subgroups, and the (normal) Sylow 3-subgroup is the unique nilpotent Fischer subgroup. On the other hand, in S_4 the Carter subgroups and the nilpotent Fischer subgroups are both the same as the Sylow 2-subgroups. In bigger soluble examples, we can get subgroups different from the Sylow (or Hall) subgroups.

Now if P is a Hall π-subgroup of G and $P \leq K \leq G$, then P is clearly a Hall π-subgroup of K; this property is called *persistence.* If further $g \in G$ then the conjugate P^g is also a Hall π-subgroup of G, and the persistence implies that P and P^g are both Hall π-subgroups of the subgroup $\langle P, P^g \rangle$ which they generate (and which we call their *join*). Finally it follows from the conjugacy property that $P^g = P^h$ for some element $h \in \langle P, P^g \rangle$. Thus Hall π-subgroups satisfy the following condition.

Definition. A subgroup H of G is *pronormal* if any 2 conjugates of H are conjugate in their join.

The argument used in the last paragraph proves that conjugacy plus persistence implies pronormality. But Carter subgroups and nilpotent Fischer subgroups both form conjugacy classes. We can also show that they are both persistent: for when $H \leq K \leq G$, the definitions imply that if H is a Carter subgroup of G then H is self-normalising in K, while if H is a nilpotent Fischer subgroup of G, then H contains every nilpotent subgroup of K which it normalises. Thus the following result is a consequence of the above theorems.

Corollary. *The Carter subgroups and nilpotent Fischer subgroups of a finite soluble group are both pronormal.*

We next define the injectors, which generalise a different property of Hall subgroups. If P is a Hall π-subgroup of G and $X \lhd G$, then $P \cap X \leq P$ so $P \cap X$ is a π-subgroup, but $|X : P \cap X| = |PX : P|$ is a factor of $|G : P|$, so $|X : P \cap X|$ is not divisible by any prime number in π. This proves that $P \cap X$ is a Hall π-subgoup of X. More generally, we say that X is a *subnormal* subgroup, and we write X sn G, if there is a chain $X = X_k \lhd \ \dots \ \lhd X_1 \lhd X_0 = G$. Then the above argument can be used inductively to show that for each of the subgroups X_i in the chain, $P \cap X_i$ is a Hall π-subgroup of X_i. We deduce that if P is a Hall π-subgroup of G and X sn G, then $P \cap X$ is a Hall π-subgroup of X. This property, which can be shown to characterise Hall subgroups, suggests the following definition and theorem.

Definition. A *nilpotent injector* of G is a subgroup H such that $H \cap X$ is a maximal nilpotent subgroup of X whenever X sn G (and in particular $H = H \cap G$ is a maximal nilpotent subgroup of G).

Theorem. (Fischer [2, VIII (4.8)]) *In a finite soluble group, the nilpotent injectors are the same as the nilpotent Fischer subgroups.*

This means that if H is a nilpotent injector, then it enjoys the same properties as a nilpotent Fischer subgroup. If also X sn G and Y sn X, then it follows from the definition of subnormality that Y sn G, and so $(H \cap X) \cap Y = H \cap Y$ is a maximal nilpotent subgroup of Y. This shows that $H \cap X$ is a nilpotent injector of X, and we get the following result.

Corollary. *Nilpotent injectors exist and are conjugate in every finite soluble group; they are also persistent, and therefore pronormal. Moreover if H is a nilpotent injector of G and X sn G, then $H \cap X$ is a nilpotent injector of X.*

To give the definition of injectors, we must replace the nilpotent subgroups of G by a more general set, as in the definition below. When $\mathfrak{F}$ is a set of subgroups of G, then the members of $\mathfrak{F}$ will be called *$\mathfrak{F}$-subgroups,* and if $X \leq G$ then $\mathfrak{F}_X$ will denote the set of $\mathfrak{F}$-subgroups which are contained in X.

Definition. A *Fitting set* in G is a set $\mathfrak{F}$ of subgroups which satisfies the following 3 conditions.

(i) Whenever $F \in \mathfrak{F}$ and $g \in G$, then $F^g \in \mathfrak{F}$ (conjugacy closure).

(ii) Whenever $F \in \mathfrak{F}$ and $E \lhd F$, then $E \in \mathfrak{F}$ (normal subgroup closure).

(iii) Whenever E and F are $\mathfrak{F}$-subgroups which normalise each other, then $EF \in \mathfrak{F}$ (normal product closure).

An *$\mathfrak{F}$-injector* of G is a subgroup H such that $H \cap X$ is a maximal $\mathfrak{F}$-subgroup of X whenever X sn G. As before it follows immediately that H is a maximal $\mathfrak{F}$-subgroup of G, and $H \cap X$ is an $\mathfrak{F}_X$-injector of X.

Theorem. (FISCHER, GASCHÜTZ & HARTLEY [2, VIII (2.9) & (2.13)]) *If $\mathfrak{F}$ is a Fitting set in a finite soluble group G, then $\mathfrak{F}$-injectors exist and are conjugate in G; they are also persistent, and therefore pronormal. Moreover if H is an $\mathfrak{F}$-injector of G and X* sn *G, then $H \cap X$ is an $\mathfrak{F}_X$-injector of X.*

It is easy to check that the π-subgroups of G form a Fitting set, so this result generalises the properties of Hall subgroups. Moreover the nilpotent subgroups of G also form a Fitting set, so we have a generalisation of the above Corollary. We note that Doerk & Hawkes [2, p. 536] speak of the proof as 'short and elegant', so the generalisation is satisfying and worthwhile. [We remark that in order to get a similar generalisation of nilpotent Fischer subgroups we must use the *Fischer sets,* which may form a proper subfamily of the family of Fitting sets.]

As our title suggests, we wish to characterise the injectors of a given finite soluble group G; by this we mean the subgroups H such that there is a Fitting set $\mathfrak{F}$ for which H is an $\mathfrak{F}$-injector of G.

Question. (DOERK & HAWKES [2, p. 553]) Which subgroups of a given finite soluble group are injectors?

As well as posing this question, Doerk and Hawkes gave a partial answer, which we now describe. It is plausible, and also true, that if H is an injector, then it should be an $\mathfrak{F}$-injector where $\mathfrak{F}$ is the set of conjugates of subnormal subgroups of H. Now this set $\mathfrak{F}$ is clearly closed under conjugacy and normal subgroups, so $\mathfrak{F}$ is a Fitting set if and only if it is closed under normal products. In this way, we get the following result.

Theorem. (DOERK & HAWKES [2, VIII (3.3)]) *If G is a finite soluble group and $H \leq G$, then the following statements are equivalent.*

(i) *The subgroup H is an injector of G.*

(ii) *The set $\{S^g : S \operatorname{sn} H,\, g \in G\}$ is a Fitting set in G.*

(iii) *Whenever S and T are subnormal subgroups of H and $g \in G$, such that S and T^g normalise each other, then ST^g is a subnormal subgroup of some conjugate of H.*

This may be regarded as analogous to the original definition of $\mathfrak{F}$-injectors. But when $\mathfrak{F}$ is the set of nilpotent subgroups, then we saw above that nilpotent injectors can also be defined as nilpotent

Fischer subgroups, so making manifest their persistence. Moreover injectors are indeed persistent, by the theorem of Fischer, Gaschütz and Hartley, so one could hope for a characterisation of injectors analogous to the definition of nilpotent Fischer subgroups. In asking the above Question, Doerk and Hawkes specified that the characterisation should be 'without recourse to the concept of a Fitting set', and their discussion indicates that they had in mind a condition involving pronormality, or something similar. Before suggesting an answer to the question, we record some properties of injectors; it is convenient to write $\operatorname{Inj} G$ for the set of injectors of G. Then the first 2 statements of the next result are corollaries of the theorem of Fischer, Gaschütz and Hartley, while the statement (iii) can be deduced from the above result of Doerk and Hawkes.

Proposition. (i) *If $H \in \operatorname{Inj} G$ and $H \leq K \leq G$, then $H \in \operatorname{Inj} K$ (persistence).*

(ii) *If $H \in \operatorname{Inj} G$ and $X \operatorname{sn} G$, then $H \cap X \in \operatorname{Inj} X$.*

(iii) *Every normal subgroup of G is an injector.*

(iv) [2, VIII (3.5)] *Every maximal subgroup of G is an injector.*

(v) [2, VIII (2.15) & (2.17)] *If $H \in \operatorname{Inj} G$ and $N \lhd G$, then $HN \in \operatorname{Inj} G$.*

(vi) [2, VIII (2.15) & (2.17)] *If $N \lhd G$ then*

$$\operatorname{Inj} G/N = \{H/N : N \leq H \in \operatorname{Inj} G\}.$$

We recall that a *chief factor* of G is a factor M/N such that $N \lhd G$ and M/N is a minimal normal subgroup of G/N. Our answer to the above Question (proved in collaboration with Arnold Feldman) is given by the following result.

Theorem. (Dark & Feldman [1]) *If G is a finite soluble group and $H \leq G$, then the following statements are equivalent.*

(i) *The subgroup H is an injector of G.*

(ii) *Whenever $g \in G$ and $X \leq G$, with $X \operatorname{sn} \langle H, X\rangle$ and $X \operatorname{sn} \langle H^g, X\rangle$, then $H \cap X$ and $H^g \cap X$ are conjugate in $\langle H \cap X, H^g \cap X\rangle$.*

(iii) *Whenever M/N is a chief factor of G, and $\overline{M} = M/N$, $\overline{H} = HN/N$, then*
either $\overline{H} \geq \overline{M}$,
or $\overline{H} \cap \overline{M} = 1$, and whenever $g \in G$ and $\overline{S} \operatorname{sn} \overline{H}$, with $\overline{S}^g \leq \overline{H}\,\overline{M}$ and $\overline{S}_1 = \overline{H} \cap \overline{S}^g\overline{M} \operatorname{sn} \overline{H}$, then $\overline{S}_1$ and $\overline{S}^g$ are conjugate in $\langle \overline{S}_1, \overline{S}^g\rangle$.

The condition (ii) is a strengthened version of pronormality, and we can prove as follows that (i) implies (ii). If $\mathfrak{F}$ is a Fitting set such that H is an $\mathfrak{F}$-injector of G, and if X sn $\langle H, X\rangle = K_1$, then H is an $\mathfrak{F}_{K_1}$-injector of K_1 by the persistence property, so it follows from the definition of $\mathfrak{F}$-injectors that $H \cap X$ is an $\mathfrak{F}_X$-injector of X. Similarly if X sn $\langle H^g, X\rangle = K_2$, then H^g is an $\mathfrak{F}_{K_2}$-injector of K_2, and hence $H^g \cap X$ is an $\mathfrak{F}_X$-injector of X. Thus $H \cap X$ and $H^g \cap X$ are both $\mathfrak{F}_X$-injectors of X, so it follows from the conjugacy and pronormality of injectors that they are conjugate in their join, as required in (ii).

The condition (iii) is technical and we shall not discuss it in detail. However we can see as follows that if $\overline{H} \cap \overline{M} = 1$, then the second alternative in (iii) follows from (ii). If $g \in G$ and $\overline{S}$ sn $\overline{H}$, with $\overline{S}^g \le \overline{H}\,\overline{M}$ and $\overline{S}_1 = \overline{H} \cap \overline{S}^g\overline{M}$ sn $\overline{H}$, then we take $\overline{X} = \overline{S}^g\overline{M}$, and we note that

$$\overline{X} = \overline{H}\,\overline{M} \cap \overline{X} = (\overline{H} \cap \overline{X})\overline{M} = \overline{S}_1\overline{M}.$$

Now $\overline{S}_1$ sn $\overline{H}$ and therefore $\overline{X} = \overline{S}_1\overline{M}$ sn $\overline{H}\,\overline{M} = \langle \overline{H}, \overline{X}\rangle$, and similarly $\overline{S}^g$ sn $\overline{H}^g$ and therefore $\overline{X} = \overline{S}^g\overline{M}$ sn $\overline{H}^g\overline{M} = \langle \overline{H}^g, \overline{X}\rangle$. It follows that there is a subgroup X such that $\overline{X} = X/N$, with X sn $\langle H, X\rangle$ and X sn $\langle H^g, X\rangle$. We can then deduce from (ii) that $H \cap X$ and $H^g \cap X$ are conjugate in their join, which implies that the images $\overline{H} \cap \overline{X}$ and $\overline{H}^g \cap \overline{X}$ are also conjugate in their join. But because $\overline{H} \cap \overline{M} = 1$, we get

$$\overline{H} \cap \overline{X} = \overline{H} \cap \overline{S}^g\overline{M} = \overline{S}_1 \quad \text{and} \quad \overline{H}^g \cap \overline{X} = \overline{H}^g \cap (\overline{S}\,\overline{M})^g = \overline{S}^g,$$

so we have shown that $\overline{S}_1$ and $\overline{S}^g$ are conjugate in their join, as required in (iii).

The deduction of (i) from (iii) is more technical. It is done by taking a minimal normal subgroup M of G and showing that we can assume inductively that $HM \in \operatorname{Inj} G$. It is then possible to combine (iii) with the above Theorem of Doerk and Hawkes to prove that H is an injector.

REFERENCES

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