Irish Math. Soc. Bulletin **53** *(2004), 35–52*

On a Class of Alternatingly Hyperexpansive Subnormal Weighted Shifts

AMEER ATHAVALE, ABHIJIT RANJEKAR AND V. M. SHOLAPURKAR

ABSTRACT. If T is a weighted shift operator on a Hilbert space with the associated weight sequence $\{\alpha_n\}_{n\geq 0}$ of positive weights, then meaningful insights into the nature of T can be gained by examining the sequence $\{\theta_n(T)\}_{n\geq 0}$ where $\theta_0(T) = 1$ and $\theta_n(T) = \Pi_{k=0}^{n-1}{\alpha_k}^2$ $(n \geq 1)$. We characterize those subnormal weighted shifts whose associated $\theta_n(T)$ are interpolated by members of a special subclass of the class of absolutely monotone functions on the non-negative real line. The special subclass has such pleasant properties as being closed under differentiation and integration. We also attempt to highlight the operator theoretic significance of such characterizations.

1. Introduction

Let $\mathcal{H}$ be a complex infinite-dimensional separable Hilbert space. By $\mathcal{B}(\mathcal{H})$ we will denote the algebra of bounded linear operators on $\mathcal{H}$. If $\{e_n\}_{n\geq 0}$ is an orthonormal basis for $\mathcal{H}$, then a weighted shift operator T on $\mathcal{H}$ with the weight sequence $\{\alpha_n = \alpha_n(T)\}_{n\geq 0}$ is defined through the relations $Te_n = \alpha_n e_{n+1}$ $(n \geq 0)$. We will always assume that $\alpha_n > 0$ for all n, and that $\{\alpha_n\}_{n\geq 0}$ is a bounded sequence so that T is in $\mathcal{B}(\mathcal{H})$. The basic properties of weighted shift operators can be found in [3] and [10]. We will use the notation $T = \Gamma(\alpha_n)$ to indicate a weighted shift with the weight sequence $\{\alpha_n\}_{n\geq 0}$. For a weighted shift $T = \Gamma(\alpha_n)$, the sequence $\{\theta_n = \theta_n(T)\}_{n\geq 0}$ is defined by $\theta_0 = 1$, $\theta_n = \prod_{k=0}^{n-1} \alpha_k^2$ $(n \geq 1)$. Note that $\theta_n = ||T^n e_0||^2$ and $\alpha_n = \sqrt{\theta_{n+1}/\theta_n}$, $n \geq 0$. An operator T in $\mathcal{B}(\mathcal{H})$ is said to be *subnormal* if there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator N in $\mathcal{B}(\mathcal{K})$ such that $N\mathcal{H} \subset \mathcal{H}$ and $N_{|\mathcal{H}} = T$.

2000 *Mathematics Subject Classification.* 47B20, 47B37, 47B39; 05A19, 11B73, 11B83.

An operator T in $\mathcal{B}(\mathcal{H})$ is said to be *alternatingly hyperexpansive* if $(-1)^n \sum_{0\leq p\leq n}(-1)^p\binom{n}{p}T^{*p}T^p \geq 0$ for all $n \geq 1$. The Bergman shift $B = \Gamma(\sqrt{\frac{n+1}{n+2}})$ is subnormal, while the Dirichlet shift $T = \Gamma(\sqrt{\frac{n+2}{n+1}})$ and the weighted shift $T = \Gamma(\frac{n+2}{n+1})$ are alternatingly hyperexpansive; any isometry is subnormal as well as alternatingly hyperexpansive. While there is a copious amount of literature on subnormal operators (refer to [3] for references), some basic facts pertaining to alternatingly hyperexpansive operators have been recorded in [1], [7] and [9].

The symbol $\mathbb{R}_+$ will denote the set $[0,\infty)$ of non-negative reals, while the symbol $\mathbb{N}$ will stand for the set of non-negative integers. For a real-valued map φ on $\mathbb{N}$ we define the (forward) difference operator Δ as follows: $(\Delta\varphi)(n) = \varphi(n) - \varphi(n+1)$. The operators Δ^n are inductively defined for all $n \geq 0$ through the relations $\Delta^0\varphi = \varphi$, $\Delta^n\varphi = \Delta(\Delta^{n-1}\varphi)(n \geq 1)$. A non-negative map φ on $\mathbb{N}$ is said to be *absolutely monotone* if $(\Delta^k\varphi)(n) \geq 0$ for all $k, n \geq 0$.

A sequence $\{\beta_n\}_{n\geq 0}$ of reals will be referred to as a *moment sequence* if there exist a positive real number b and a positive Borel measure μ on [0,b] such that

$$\beta_n = \int_{[0,b]} x^n d\mu(x) \qquad (n \geq 0). \tag{1}$$

It is well known that a weighted shift $T = \Gamma(\alpha_n)$ is subnormal if and only if the sequence $\{\theta_n(T)\}_{n\geq 0}$ is a moment sequence corresponding to a probability measure on $\left[0, ||T||^2\right]$ (refer to [4]).

For x in $\mathbb{R}_+$ and k in $\mathbb{N}$, let $(x)_k$ denote the "falling factorial of x", that is, $(x)_0 = 1$ and $(x)_k = x(x-1)\ldots(x-k+1)$ for $k \geq 1$. Given a sequence $\{\beta_n\}_{n\geq 0}$ of reals, we have by Newton's Interpolation Formula, $\beta_n = (1+\triangle)^n\beta_0$, that is,

$$\beta_n = \sum_{k=0}^{n} a_k(n)_k, \tag{2}$$

where $a_k = \triangle^k\beta_0/k!$

It was noted in [9] that a sequence $\{\beta_n\}_{n\geq 0}$ of reals is absolutely monotone if and only if $a_k \geq 0$ for all k in (2). Further, as was recorded in [9], a weighted shift $T = \Gamma(\alpha_n)$ is alternatingly hyperexpansive if and only if the sequence $\{\theta_n(T)\}_{n\geq 0}$ is absolutely monotone.

A continuous non-negative function on $\mathbb{R}_+$ is said to be *absolutely monotone* (*on* $\mathbb{R}_+$) if f is infinitely differentiable on $(0,\infty)$ and satisfies $f^{(n)}(s) \geq 0$ at all points s of $(0,\infty)$ and for all $n \geq 0$. It is well known that an absolutely monotone function has a power series representation on $\mathbb{R}_+$ (refer to [11]). A sequence $\{\beta_n\}_{n\geq 0}$ is said to be *interpolated* by a function f defined on $\mathbb{R}_+$ if $f(n) = \beta_n$ for all n in $\mathbb{N}$. While it is known that any sequence interpolated by an absolutely monotone function is absolutely monotone (see [11], for example), it was observed in [9] that not every absolutely monotone sequence arises that way.

In the sequel, it will be convenient to use the symbol AH to refer to the class of alternatingly hyperexpansive operators (on a given Hilbert space), and the symbols $\mathcal{A}MS$ and $\mathcal{A}MF$, respectively to refer to the class of absolutely monotone sequences and the class of absolutely monotone functions on $\mathbb{R}_+$, respectively. The intersection of the class AH with the class of subnormals defies an easy description. Indeed, as was shown in Proposition 4.1 of [9], if a subnormal operator T is such that the spectrum of its 'minimal' normal extension is contained in the complement of the open unit disk in the complex plane centered at the origin, then T is alternatingly hyperexpansive; on the other hand there exist alternatingly hyperexpansive subnormal operators such that the spectrum of their minimal normal extension intersects the open unit disk—consider, for example, the weighted shift $T = \Gamma(\sqrt{\cosh(n+1)/\cosh(n)})$ for which $\{\theta_n(T) = \cosh(n)\}_{n\geq 0}$ is a moment sequence with the corresponding measure μ concentrated at the points e and $1/e$. In the context of weighted shifts, and in view of our discussion above, an almost tautological answer to the problem under consideration is: A weighted shift T is subnormal as well as in AH if and only if $\{\theta_n(T)\}_{n\geq 0}$ is an absolutely monotone moment sequence. A natural question to raise, then, is: Can one characterize those subnormal weighted shifts T whose associated $\{\theta_n(T)\}_{n\geq 0}$ are interpolated by members of $\mathcal{A}MF$? While the question could be of interest in its own right, Theorem 1 below provides a strong motivation for examining such sequences; indeed, the statement of Theorem 1 is the statement of a 'spectral permanence property' of T under the operations of differentiation and integration on the corresponding member of $\mathcal{A}MF$.

As has been pointed out in [2], there is not much literature on absolutely monotone sequences. To the knowledge of the authors, there is no known characterization of absolutely monotone sequences interpolated by absolutely monotone functions; to that extent, the question posed above appears difficult to answer. One could, however, hope to obtain some positive results by looking at special subclasses of $\mathcal{AMF}$. In Section 2 we introduce a special subclass $\mathcal{AMF}^*$ of $\mathcal{AMF}$, and in Section 3 we provide an explicit characterization of those moment sequences that are interpolated by members of $\mathcal{AMF}^*$. The class $\mathcal{AMF}^*$ enjoys several pleasant properties; in particular, we establish that $\mathcal{AMF}^*$ is closed under the operations of differentiation and integration and allows the action of differentiation to be interpreted as $\log(1+\Delta)$.

We conclude the present section by attempting to highlight the operator theoretic motivation underlying the considerations in Sections 2 and 3.

If $T = \Gamma(\alpha_n)$ is a weighted shift operator such that $\lim\limits_{n\to\infty} \alpha_n = \alpha$ exists, then it is well known (see [3]) that the assertions (S1), (S2) and (S3) below hold:

(S1) The spectrum $\sigma(T)$ of T equals $\{\lambda : \mid \lambda \mid \leq \alpha\}$.

(S2) The essential spectrum $\sigma_e(T)$ of T equals $\{\lambda : \mid \lambda \mid = \alpha\}$.

(S3) If $\mid \lambda \mid < \alpha$, then $T - \lambda I$ is Fredholm with the Fredholm index $\mathrm{ind}(T-\lambda I)$ of $T-\lambda I$ being equal to -1.

Lemma 1. *Suppose $f : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$ is such that $f(s) = \int_{(0,b]} x^s d\mu(x)$ with μ a positive Borel measure and b a positive real. Then $\lim\limits_{s\to\infty} \frac{f(s+1)}{f(s)}$ exists and equals the μ-essential sup norm $||x||_\infty$ of x.*

Proof. With f as above, we have $\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = ||x||_\infty$ (see [8]). Let $[s]$ denote the integral part of s and let

$$g(s) = \int_{(0,b]} (x/||x||_\infty)^{[s]} d\mu(x) \qquad (s \geq 0).$$

Using $[s] \leq s < [s]+1$, $[s+n] = [s]+n$ $(n \in \mathbb{N})$, and $x/||x||_\infty \leq 1$ ($[\mu]$-a.e.), we have

$$||x||_\infty \frac{g(s+2)}{g(s)} \leq \frac{f(s+1)}{f(s)} \leq ||x||_\infty \frac{g(s)}{g(s+1)} \ (s \geq 0). \tag{3}$$

Let now

$$g_1(s) = \frac{g(s+1)}{g(s)} = \frac{1}{||x||_\infty}\frac{f([s]+1)}{f([s])},\ s \geq 0.$$

Clearly, $g_1(s) = \dfrac{1}{||x||_\infty}\dfrac{f(n+1)}{f(n)}$ for $n \leq s < n+1$, so that

$$\lim_{s\to\infty} g_1(s) = \frac{1}{||x||_\infty}\lim_{n\to\infty}\frac{f(n+1)}{f(n)} = 1.$$

Appealing to the inequalities in (3) we arrive at the assertion $\lim\limits_{s\to\infty}\dfrac{f(s+1)}{f(s)} = ||x||_\infty$. □

Theorem 1. *Let $T = \Gamma(\alpha_n)$ be a weighted shift operator which is both subnormal and alternatingly hyperexpansive, and such that the sequence $\{\theta_n(T)\}_{n\geq 0}$ is interpolated by an absolutely monotone function $f(x) = \sum_{j\geq 0} b_j x^j$ $(x \geq 0)$ satisfying $f(0) = 1$ and $f(s) = \int_{(0,b]} x^s d\mu(x)$ $(s > 0)$ with μ a positive Borel measure and b a positive real. Let T^d and T^i be the weighted shift operators with $\theta_n(T^d) = \frac{1+f'(n)}{1+f'(0)}$ and $\theta_n(T^i) = F(n)$ where $F(x) = 1+\sum_{j\geq 0}\frac{b_j}{j+1}x^{j+1}$ $(x \geq 0)$. If α is the μ-essential sup norm $||x||_\infty$ of x, then all of T, T^d and T^i satisfy* (S1)*,* (S2) *and* (S3)*.*

Proof. With T as above we have, from the well-known property of subnormals, that $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\frac{f(n+1)}{f(n)}$ exists; from our observations in Lemma 1 it is clear that this limit equals $\alpha = ||x||_\infty$, and that $\lim_{s\to\infty}\frac{f(s+1)}{f(s)} = \alpha$. While the desired conclusion for T is now obvious, that for T^i follows from noting that $F'(s) = f(s)$ and applying L'Hopital's Rule. Since $f''(s) = \int_{(0,b]} x^s(\log x)^2 d\mu(x)$, we have by Lemma 1 again that $\beta = \lim_{s\to\infty}\frac{f''(s+1)}{f''(s)}$ exists. In view of L'Hopital's Rule, $\lim_{s\to\infty}\frac{1+f'(s+1)}{1+f'(s)}$ and $\lim_{s\to\infty}\frac{f'(s+1)}{f'(s)}$ must exist and equal β. Yet another application of L'Hopital's Rule shows that β equals α and the desired conclusion regarding T^d follows. □

The weighted shift operator $T = \Gamma(\sqrt{\cosh(n+1)/\cosh(n)})$ satisfies the hypotheses of Theorem 1 so that T and $T^d = T^i = \Gamma(\sqrt{(1+\sinh(n+1))/(1+\sinh(n))})$ satisfy (S1), (S2) and (S3); it should be noted that $T^d = T^i = \Gamma(\sqrt{(1+\sinh(n+1))/(1+\sinh(n))})$ is not subnormal.

2. The Subclass $\mathcal{AMF}^*$ of $\mathcal{AMF}$

In the sequel we require a number of combinatorial properties of *Stirling numbers $S_1(n,k)$ of the first kind* and *Stirling numbers $S_2(n,k)$ of the second kind* $(n,k \in \mathbb{N})$; $S_1(n,k)$ and $S_2(n,k)$ may formally be defined through relations (D1) and(D2) below as considered valid for any real number x. For the basic information on Stirling numbers, the reader is referred to [5]. Some elementary facts regarding Stirling numbers are: $S_1(0,0) = S_2(0,0) = 1$; $S_1(n,0) = S_2(n,0) = 0$ $(n \geq 1)$; $S_1(n,1) = (n-1)!$ $(n \geq 1)$; $S_2(n,1) = 1$ $(n \geq 1)$; $S_2(n,2) = 2^{n-1} - 1$ $(n \geq 1)$; $S_1(n,n) = S_2(n,n) = 1$ $(n \geq 0)$; $S_1(n,k) = S_2(n,k) = 0$ $(n < k)$.

Further, we require recurrence relations (R1), (R2) (with n,k in $\mathbb{N} \setminus \{0\}$), inversion formulas (I1), (I2), equalities (E1), (E2), (E3), and "generating function formulas" (G1), (G2) (with n,k,m in $\mathbb{N}$) as given below and for which the reader is referred to Chapters 6 and 7 of [5].

(D1) $(x)_n = \sum_{k=0}^{n} S_1(n,k)(-1)^{n-k} x^k$

(D2) $x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k$

(R1) $S_1(n,k) = (n-1)S_1(n-1,k) + S_1(n-1,k-1)$

(R2) $S_2(n,k) = kS_2(n-1,k) + S_2(n-1,k-1)$

(I1) $\sum_{k\geq 0} S_1(n,k)S_2(k,m)(-1)^{n-k} = \delta_{n,m}$

(I2) $\sum_{k\geq 0} S_2(n,k)S_1(k,m)(-1)^{n-k} = \delta_{n,m}$

(E1) $S_2(n+1,m+1) = \sum_{k\geq 0} \binom{n}{k} S_2(k,m)$

(E2) $m!S_2(n,m) = \sum_{k\geq 0} \binom{m}{k} k^n (-1)^{(m-k)}$

(E3) $\binom{n}{m} = \sum_{k\geq 0} S_2(n+1,k+1)S_1(k,m)(-1)^{(m-k)}$

(G1) $\left(\log \frac{1}{1-x}\right)^m = m! \sum_{k\geq 0} S_1(k,m) \frac{x^k}{k!} \quad (-1 < x < 1)$

(G2) $\frac{1}{(1-x)^w} = \sum_{k\geq 0} \sum_{r\geq 0} S_1(k,r) w^r \frac{x^k}{k!} \quad (-1 < x < 1, w \in \mathbb{R}_+)$.

Given a sequence $\{\gamma_n = \sum_{k=0}^{n} a_k (n)_k\}_{n\geq 0}$ of reals, we define, for $r \geq 0$,

$$b_r = \sum_{k\geq 0} (-1)^{k-r} a_k S_1(k,r).$$

We use $\mathcal{AMS}^*$ to denote the set of those sequences $\{\gamma_n\}_{n\geq 0}$ in $\mathcal{AMS}$ that satisfy conditions (A) and (B) below:

(A) $\sum_{k\geq 0}\sum_{r\geq 0} a_k S_1(k,r)n^r < \infty$ for each $n \in \mathbb{N}$ (here 0^0 is interpreted as the number 1);

(B) $b_r \geq 0$.

Further, we use $\mathcal{AMF}^*$ to denote the set of those functions $f(x) = \sum_{r\geq 0} c_r x^r$ in $\mathcal{AMF}$ for which

(C) $\sum_{j\geq 0}\sum_{k\geq 0}\sum_{r\geq 0} c_j S_2(j,k)S_1(k,r)n^r < \infty$ for each $n \in \mathbb{N}$.

Theorem 2. *Every sequence $\{\gamma_n\}_{n\geq 0}$ in $\mathcal{AMS}^*$ is interpolated by a member of $\mathcal{AMF}^*$. Conversely, any sequence $\{\gamma_n\}_{n\geq 0}$ that is interpolated by a member of $\mathcal{AMF}^*$ is a member of $\mathcal{AMS}^*$.*

Proof. If $\{\gamma_n\}_{n\geq 0}$ is in $\mathcal{AMS}^*$, then it follows from condition (A) that $\sum_{r\geq 0} b_r n^r < \infty$ for all $n \geq 0$. Consider the function $f(x) = \sum_{r\geq 0} b_r x^r$ on $\mathbb{R}_+$. Clearly f is a convergent power series on $\mathbb{R}_+$ and condition (B) shows that f is an absolutely monotone function. Further, $f(n) = \sum_{r\geq 0} b_r n^r = \sum_{r\geq 0}[\sum_{k\geq 0}(-1)^{k-r} a_k S_1(k,r)]n^r = \sum_{k\geq 0} a_k[\sum_{r\geq 0}(-1)^{k-r} S_1(k,r)n^r] = \sum_{k\geq 0} a_k (n)_k = \gamma_n$ (the interchange of summations is justified by condition (A)). Hence f is an absolutely monotone function that interpolates the sequence $\{\gamma_n\}_{n\geq 0}$. Further, $\gamma_n = f(n) = \sum_{r\geq 0} b_r n^r = \sum_{r\geq 0} b_r[\sum_{k\geq 0} S_2(r,k)(n)_k] = \sum_{k\geq 0}[\sum_{r\geq 0} S_2(r,k)b_r](n)_k$ so that $a_k = \sum_{r\geq 0} S_2(r,k)b_r$. Also,

$$\begin{aligned}\sum_{j\geq 0}\sum_{k\geq 0}\sum_{r\geq 0} b_j S_2(j,k)S_1(k,r)n^r \\ &= \sum_{r\geq 0}\sum_{k\geq 0}[\sum_{j\geq 0} b_j S_2(j,k)]S_1(k,r)n^r \\ &= \sum_{r\geq 0}\sum_{k\geq 0} a_k S_1(k,r)n^r < \infty\end{aligned}$$

by condition (A). Hence the function f interpolating the sequence $\{\gamma_n\}_{n\geq 0}$ satisfies condition (C) and is a member of $\mathcal{AMF}^*$.

Conversely, for $f = \sum_{k\geq 0} c_k x^k$ in $\mathcal{AMF}^*$, consider the absolutely monotone sequence $\{\gamma_n = f(n) = \sum_{k=0}^{n} a_k (n)_k \}_{n\geq 0}$ interpolated by f. From our observations above it follows that $a_k =$

$\sum_{r\geq 0} S_2(r,k)c_r$. Now,

$$\begin{aligned} b_j &= \sum_{k\geq 0}(-1)^{k-j}a_k S_1(k,j) \\ &= \sum_{k\geq 0}\Big[\sum_{r\geq 0} S_2(r,k)c_r\Big](-1)^{k-j}S_1(k,j). \end{aligned}$$

Since the function f is a member of $\mathcal{A}MF^*$, the series representing b_j is absolutely convergent so that we have

$$\begin{aligned} b_j &= \sum_{r\geq 0}\Big[\sum_{k\geq 0}(-1)^{k-r}S_2(r,k)S_1(k,j)\Big](-1)^{r-j}c_r \\ &= \sum_{r\geq 0}\delta(r,j)(-1)^{r-j}c_r = c_j. \end{aligned}$$

Thus the sequence $\{\gamma_n = f(n)\}_{n\geq 0}$ satisfies condition (B). Further,

$$\begin{aligned} &\sum_{k\geq 0}\sum_{r\geq 0} a_k S_1(k,r)n^r \\ &\qquad = \sum_{k\geq 0}\sum_{r\geq 0}\sum_{p\geq 0} S_2(p,k)c_p S_1(k,r)n^r < \infty, \end{aligned}$$

since the function f is a member of $\mathcal{A}MF^*$. Thus the sequence $\{\gamma_n = f(n)\}_{n\geq 0}$ satisfies condition (A) as well. $\square$

Examples 1. (i) Any sequence $\{\beta_n\}_{n\geq 0}$ for which $\beta_n = p(n)$, where $p(n)$ is a polynomial with non-negative coefficients, is a member of $\mathcal{A}MS^*$. If, moreover, $p(0) = 1$, then $p(n)$ may be looked upon as $\theta_n(T)$ corresponding to a weighted shift T that is an alternatingly hyperexpansive d-isometry (refer to [9]). Besides the trivial case of the constant polynomial $p(z) = 1$, the weighted shifts so obtained are necessarily non-subnormal.

(ii) For the sequence $\{\beta_n = (1+\delta)^n\}_{n\geq 0}$, with $0 \leq \delta < 1$, we have $a_k = \delta^k/k!$. Since $0 \leq \delta < 1$, we have, in view of identity (G2), $\sum_{k\geq 0}\sum_{r\geq 0} a_k S_1(k,r)n^r = \sum_{k\geq 0}\sum_{r\geq 0} S_1(k,r)\frac{\delta^k}{k!}n^r = \frac{1}{(1-\delta)^n} < \infty$ for all $n \geq 0$. Hence $\{\beta_n\}_{n\geq 0}$ satisfies condition (A).

Further, $b_r = \sum_{k\geq 0}(-1)^{k-r}S_1(k,r)a_k = \sum_{k\geq 0}(-1)^{k-r}S_1(k,r)\frac{\delta^k}{k!}$. Since $0 \leq \delta < 1$, we have $b_r = \frac{1}{r!}[\log(1+\delta)]^r$ $(r \geq 0)$, in view of identity (G1). Thus $\{\beta_n\}_{n\geq 0}$ satisfies condition (B) as well. It is seen that $\{\beta_n\}_{n\geq 0}$ is interpolated by the absolutely monotone function

$f(x) = \sum_{r\geq 0} b_r x^r = \sum_{r\geq 0} \frac{1}{r!}[\log(1+\delta)]^r x^r = e^{x\log(1+\delta)} = (1+\delta)^x$. We will return to this example following Theorem 6 below.

Theorems 3 and 5 below show that the class $\mathcal{A}MF^*$ is closed under the operations of differentiation and integration, while Theorem 4 below deciphers the action of differentiation on that class.

Theorem 3. *If f belongs to $\mathcal{A}MF^*$ then the derivatives $f^{(k)}$ belong to $\mathcal{A}MF^*$ for all $k \geq 0$.*

Proof. If $f(x) = \sum_{j\geq 0} b_j x^j$ then $f'(x) = \sum_{j\geq 0} c_j x^j$, where $c_j = (j+1)b_{j+1}$. In view of identity (E1) we have

$$(j+1)S_2(j,k) \leq S_2(j+2,k+1).$$

Thus

$$\begin{aligned}\sum_{j\geq 0}\sum_{k\geq 0}\sum_{r\geq 0}&(j+1)b_{j+1}S_2(j,k)S_1(k,r)n^r \\ &\leq \sum_{j\geq 0}\sum_{k\geq 0}\sum_{r\geq 0} b_{j+1}S_2(j+2,k+1)S_1(k,r)n^r \\ &= P_1 + P_2 + P_3,\end{aligned}$$

where

$$\begin{aligned} P_1 &= \sum_{j\geq 0}\sum_{k\geq 0}\sum_{r\geq 0} b_{j+1}S_2(j+1,k+1)kS_1(k,r)n^r, \\ P_2 &= \sum_{j\geq 0}\sum_{k\geq 0}\sum_{r\geq 0} b_{j+1}S_2(j+1,k+1)S_1(k,r)n^r \end{aligned}$$

and

$$P_3 = \sum_{j\geq 0}\sum_{k\geq 0}\sum_{r\geq 0} b_{j+1}S_2(j+1,k)S_1(k,r)n^r.$$

(Here we used the recurrence relation (R2) for $S_2(j+2,k+1)$.) Since f is a member of $\mathcal{A}MF^*$, we have $P_3 < \infty$. The finiteness of P_1 and that of P_2 follow by noting that $kS_1(k,r) \leq S_1(k+1,r)$ and $S_1(k,r) \leq S_1(k+1,r+1)$ and using condition (C) for the function f. Thus f' satisfies condition (C) and hence belongs to $\mathcal{A}MF^*$. $\square$

The difference operator $\triangle$ can be interpreted to act on any function f defined on $\mathbb{R}_+$ through the relation $\triangle f(x) = f(x+1) - f(x)$. Interpreted thus, $\triangle$ acts on any real analytic function f as $e^D - 1$, where D denotes differentiation (refer to [5], Chapter 9). It is further known (refer to [6], Chapter 5) that D acts on polynomials as $\log(1+\triangle)$ (see Definition 1 below). In Theorem 4 below, we establish that D effectively acts as $\log(1+\triangle)$ on any member of the class $\mathcal{A}MF^*$.

Lemma 2. (a) *If* $\{\gamma_n = \sum_{k=0}^{n} a_k (n)_k\ \}_{n\geq 0}$ *is in* $\mathcal{A}MS^*$*, then* $\sum_{k\geq 0} \frac{\triangle^k \gamma_d}{k!} S_1(k,r) < \infty$ *for all non-negative integers* d *and* r*.*
(b) *If* f *is in* $\mathcal{A}MF^*$*, then* $\sum_{p\geq 0}\sum_{k\geq 0} \frac{f^p(d)}{p!} S_2(p,k) S_1(k,r) < \infty$ *for all non-negative integers* d *and* r*.*

Proof. (a) For the sequence $\{\tilde{\gamma_n} = \gamma_{n+1} = \sum_{k=0}^{n} \tilde{a}_k (n)_k\}$, we have $\tilde{a_k} = \triangle^k \tilde{\gamma_0}/k! = \triangle^k \gamma_1/k! = \triangle^k (1+\triangle)\gamma_0/k! = \triangle^k \gamma_0/k! + \triangle^{k+1}\gamma_0/k! = a_k + (k+1)a_{k+1}$. Consider

$$\begin{aligned}\sum_{k\geq 0}\sum_{r\geq 0} \tilde{a_k} S_1(k,r) n^r &= \sum_{k\geq 0}\sum_{r\geq 0} [a_k + (k+1)a_{k+1}] S_1(k,r) n^r \\ &= P_1 + P_2 + P_3,\end{aligned}$$

$P_1 = \sum_{k\geq 0}\sum_{r\geq 0} a_k S_1(k,r) n^r$, $P_2 = \sum_{k\geq 0}\sum_{r\geq 0} k a_{k+1} S_1(k,r) n^r$ and $P_3 = \sum_{k\geq 0}\sum_{r\geq 0} a_{k+1} S_1(k,r) n^r$. The finiteness of P_1 follows from condition (A). Using the recurrence relation (R1), one has $kS_1(k,r) \leq S_1(k+1,r)$, leading to the finiteness of P_2. Further, using the same recurrence relation, one has $S_1(k,r) \leq S_1(k+1,r+1)$, leading to the finiteness of P_3. Thus $\{\tilde{\gamma_n}\}$ satisfies condition (A). Since $\{\gamma_n\}_{n\geq 0}$ is in $\mathcal{A}MS^*$, there exists a function $f(x) = \sum_{p\geq 0} b_p x^p$ in $\mathcal{A}MF^*$ such that $\gamma_n = f(n)$ for $n \in \mathbb{N}$. Hence,

$$\begin{aligned}\triangle^k \tilde{\gamma_0} = \triangle^k \gamma_1 = \triangle^k f(1) &= \sum_{r=0}^{k} \binom{k}{r} (-1)^{k-r} f(1+r) \\ &= \sum_{r=0}^{k} \binom{k}{r} (-1)^{k-r} \Big[\sum_{p\geq 0} b_p (1+r)^p\Big] \\ &= \sum_{p\geq 0} \Big[\sum_{r=0}^{k} \binom{k}{r} (-1)^{k-r} (1+r)^p\Big] b_p \\ &= \sum_{p\geq 0} d(k,p) b_p.\end{aligned}$$

Now, by (E2),

$$\begin{aligned}
d(k,p) &= \sum_{r=0}^{k} \binom{k}{r} (-1)^{k-r}(1+r)^p \\
&= \sum_{r=0}^{k} \frac{1+r}{k+1}\binom{k+1}{r+1}(-1)^{k-r}(1+r)^p \\
&= \frac{1}{k+1}\sum_{r=0}^{k}\binom{k+1}{r+1}(-1)^{k-r}(1+r)^{p+1} \\
&= \frac{1}{k+1}(k+1)!\,S_2(p+1,k+1) \\
&= k!\,S_2(p+1,k+1).
\end{aligned}$$

Hence, $\tilde{a_k} = \triangle^k \tilde{\gamma_0}/k! = \sum_{p\geq 0} d(k,p)b_p/k! = \sum_{p\geq 0} S_2(p+1,k+1)b_p$. Thus,

$$\begin{aligned}
\tilde{b_r} &\equiv \sum_{k\geq 0} S_1(k,r)(-1)^{k-r}\tilde{a_k} \\
&= \sum_{k\geq 0} S_1(k,r)(-1)^{k-r}\Big[\sum_{p\geq 0} S_2(p+1,k+1)b_p\Big] \\
&= \sum_{p\geq 0}\Big[\sum_{k\geq 0} S_2(p+1,k+1)S_1(k,r)(-1)^{k-r}\Big]b_p \\
&= \sum_{p\geq 0}\binom{p}{r}b_p,
\end{aligned}$$

where in the last step we used identity (E3). Since $\sum_{p\geq 0}\binom{p}{r}b_p = f^r(1)/r!$, we have $0 \leq \tilde{b_r} < \infty$. (The justification for the interchange of summations can be provided as follows:

$$\begin{aligned}
&\sum_{p\geq 0}\sum_{k\geq 0} S_2(p+1,k+1)S_1(k,r)b_p \\
&\qquad = \sum_{p\geq 0}\sum_{k\geq 0}[S_2(p,k)+(k+1)S_2(p,k+1)]S_1(k,r)b_p \\
&\qquad = Q_1 + Q_2 + Q_3,
\end{aligned}$$

where

$$Q_1 = \sum_{p\geq 0}\sum_{k\geq 0} S_2(p,k)S_1(k,r)b_p,$$
$$Q_2 = \sum_{p\geq 0}\sum_{k\geq 0} S_2(p,k+1)kS_1(k,r)b_p,$$
$$Q_3 = \sum_{p\geq 0}\sum_{k\geq 0} S_2(p,k+1)S_1(k,r)b_p.$$

Since f is in $\mathcal{A}MF^*$, using condition (C) for $n=1$, the finiteness of Q_1 follows. The finiteness of Q_2 and that of Q_3 follow by appealing to the recurrence relations (R1) and (R2) again.) Thus $\{\tilde{\gamma_n}\}$ satisfies condition (B) as well. Hence $\{\tilde{\gamma_n}\}$ is a member of $\mathcal{A}MS^*$. It follows that the sequence $\{\gamma_{n+d}\}_{n\geq 0}$ is in $\mathcal{A}MS^*$ for every $d \geq 0$. In particular, $\sum_{k\geq 0}\sum_{r\geq 0}\frac{\triangle^k\gamma_d}{k!}S_1(k,r)n^r < \infty$ for every $d\geq 0$. Putting $n=1$, the desired conclusion follows.

(b) If the sequence $\{\gamma_n\}_{n\geq 0}$ belongs to $\mathcal{A}MS^*$, then it follows from the proof of Theorem 1 that the function $f(x) = \sum_{r\geq 0} b_r x^r$ belongs to $\mathcal{A}MF^*$. By part (a) above, the sequence $\{\tilde{\gamma_n}\}$ belongs to $\mathcal{A}MS^*$ and hence the function $\tilde{f}(x) = \sum_{r\geq 0}\tilde{b}_r x^r$ is a member of $\mathcal{A}MF^*$. As shown in the proof of part (a) above, we have $\tilde{b}_r = f^r(1)/r!$; thus $\tilde{f}(x) = \sum_{r\geq 0} f^r(1)/r! x^r = f(x+1)$. This shows that $g_1(x) = f(x+1)$ is a member of $\mathcal{A}MF^*$, and so is $g_d(x) = f(x+d)$ for every integer $d\geq 0$. Thus g_d satisfies condition (C), that is, $\sum_{p\geq 0}\sum_{k\geq 0}\sum_{r\geq 0}\frac{f^p(d)}{p!}S_2(p,k)S_1(k,r)n^r < \infty$ for every $n\geq 0$. Putting $n=1$, the desired conclusion follows. □

For $m\in\mathbb{N}$, let $\mathcal{A}MS^{(m)}$ denote the class of absolutely monotone sequences $\{\gamma_n\}_{n\geq 0}$ for which $\sum_{p\geq m} m!(-1)^{p-m}S_1(p,m)\frac{\triangle^p\gamma_n}{p!}$ are finite real numbers for all $n\geq 0$.

Definition 1. Given a sequence $\{\gamma_n\}_{n\geq 0}$ in $\mathcal{A}MS^{(m)}$, define

$$[\log(1+\triangle)]^m\gamma_n = \sum_{p\geq m} m!(-1)^{p-m}S_1(p,m)\frac{\triangle^p\gamma_n}{p!}.$$

(In particular, for $m=1$,

$$\log(1+\triangle)\gamma_n = \sum_{k\geq 1}\frac{(-1)^{k+1}}{k}\triangle^k\gamma_n = \sum_{k\geq 1}(-1)^{k-1}S_1(k,1)\frac{\triangle^k\gamma_n}{k!}.)$$

Theorem 4. *The inclusion $\mathcal{AMS}^* \subset \mathcal{AMS}^{(m)}$ holds for every positive integer m; moreover, the following commutative diagram holds with D denoting differentiation:*

$$\begin{tikzcd} \mathcal{AMS}^* \arrow[r, "{[\log(1+\triangle)]^m}"] & \mathcal{AMS}^* \\ \mathcal{AMF}^* \arrow[u, "I"'] \arrow[r, "D^m"] & \mathcal{AMF}^* \arrow[u, "I"'] \end{tikzcd}$$

Proof. Suppose a sequence $\{\gamma_n\}_{n\geq 0}$ is interpolated by f in $\mathcal{AMF}^*$; then $\{\gamma_n\}_{n\geq 0}$ is a member of $\mathcal{AMS}^*$ by Theorem 1, and by part (a) of Lemma 2 we have that $\sum_{p=0}^{\infty} \frac{\triangle^p \gamma_n}{p!} S_1(p,m) < \infty$ for every $n \geq 0$ and $m \geq 0$. Hence $[\log(1+\triangle)]^m \gamma_n$ is a finite real number for every $m \geq 0$ and $n \geq 0$. Since $\gamma_n = f(n)$ for all $n \geq 0$, we have

$$\begin{aligned} \triangle^p \gamma_n = \triangle^p f(n) &= (-1)^p \sum_{r=0}^{p} \binom{p}{r} (-1)^r f(n+r) \\ &= (-1)^p \sum_{r=0}^{p} \binom{p}{r} (-1)^r \Big[\sum_{k=0}^{\infty} r^k f^{(k)}(n)/k!\Big] \\ &= \sum_{k=0}^{\infty} \Big[\sum_{r=0}^{p} \binom{p}{r} (-1)^{p-r} r^k\Big] f^{(k)}(n)/k! \\ &= \sum_{k=0}^{\infty} p!\, S_2(k,p) f^{(k)}(n)/k!, \end{aligned}$$

where in the last step we used identity (E2). Also,

$$\begin{aligned} [\log(1+\triangle)]^m \gamma_n &= \sum_{p=0}^{\infty} m!(-1)^{p-m} S_1(p,m) \frac{\triangle^p \gamma_n}{p!} \\ &= \sum_{p=0}^{\infty} m!(-1)^{p-m} S_1(p,m) \Big[\sum_{k=0}^{\infty} S_2(k,p) f^{(k)}(n)/k!\Big] \end{aligned}$$

$$
\begin{aligned}
&= m! \sum_{k=0}^{\infty} \Big[\sum_{p=0}^{\infty} S_2(k,p) S_1(p,m) (-1)^{k-p} \Big] (-1)^{m-k} f^{(k)}(n)/k! \\
&= m! \sum_{k=0}^{\infty} \delta_{k,m} (-1)^{m-k} f^{(k)}(n)/k! \\
&= m! f^{(m)}(n)/m! = f^{(m)}(n).
\end{aligned}
$$

(The interchange of summations above is justified by part (b) of Lemma 2.) □

Theorem 5. *If $f(x) = \sum_{j\geq 0} b_j x^j$ belongs to $\mathcal{A}MF^*$, then the function $F(x) = 1 + \sum_{j\geq 0} \frac{b_j}{j+1} x^{j+1}$ (obtained by integrating the series for $f(x)$ term by term) belongs to $\mathcal{A}MF^*$.*

Proof. If $F(x) = \sum_{j\geq 0} \tilde{b_j} x^j$ then $\tilde{b_0} = 1, \tilde{b_j} = \frac{b_{j-1}}{j}$ $(j \geq 1)$. Using the recurrence relation (R2) we have

$$
\sum_{j\geq 0} \sum_{k\geq 0} \sum_{r\geq 0} \tilde{b_j} S_2(j,k) S_1(k,r) n^r = 1 + P_1 + P_2,
$$

where $P_1 = \sum_{j\geq 1} \sum_{k\geq 0} \sum_{r\geq 0} \frac{b_{j-1}}{j} k S_2(j-1,k) S_1(k,r) n^r$ and $P_2 = \sum_{j\geq 1} \sum_{k\geq 0} \sum_{r\geq 0} \frac{b_{j-1}}{j} S_2(j-1,k-1) S_1(k,r) n^r$. For $k > j$ one has $S_2(j,k) = 0$ so that in P_1 one may only consider the terms for which $k \leq j$. Thus

$$
P_1 \leq \sum_{j\geq 1} \sum_{k\geq 0} \sum_{r\geq 0} b_{j-1} S_2(j-1,k) S_1(k,r) n^r.
$$

Further, it follows from the recurrence relation (R1) that $S_1(k,r) \leq (k-1) S_1(k-1,r-1)$. Thus

$$
P_2 \leq \sum_{j\geq 1} \sum_{k\geq 0} \sum_{r\geq 0} b_{j-1} S_2(j-1,k-1) S_1(k-1,r-1) n^r.
$$

The finiteness of P_1 and that of P_2 follow by using condition (C) for the function f. Thus F satisfies condition (C) and hence belongs to $\mathcal{A}MF^*$. □

3. MOMENT SEQUENCES IN $\mathcal{A}MS^*$

Theorem 6 below provides necessary and sufficient conditions for an absolutely monotone moment sequence to fall in $\mathcal{A}MS^*$.

Theorem 6. *Let* $\{\beta_n = \sum_{k\geq 0} a_k(n)_k\}_{n\geq 0}$ *be an absolutely monotone sequence as well as a moment sequence with the representation* $\beta_n = \int_{[0,b]} x^n d\mu(x)$ $(n \geq 0)$ (*where* μ *is a finite positive Borel measure on* $[0,b]$, $b > 0$). *Then* $\{\beta_n\}_{n\geq 0}$ *is in* $\mathcal{AMS}^*$ *if and only the measure* μ *is supported on the open interval* $(0,2)$ *and satisfies conditions* (A*) *and* (B*) *below:*

(A*) $\int_{[1,2)} \frac{1}{(2-x)^n} d\mu(x) < \infty$ *for every positive integer* n.

(B*) $\int_{(0,2)} (\log x)^r d\mu(x) \geq 0$ *for every odd positive integer* r.

Proof. For any sequence $\{\beta_n\}_{n\geq 0}$ as in the hypotheses of the theorem and with μ supported on $(0,2)$ in particular, one may write $a_k = A_k + (-1)^k B_k$, where $A_k = \int_{[1,2)} \frac{(x-1)^k}{k!} d\mu(x)$ for $k \geq 0$ and $B_k = \int_{(0,1)} \frac{(1-x)^k}{k!} d\mu(x)$ for $k \geq 0$. Since $a_k \geq 0$ for all $k \geq 0$, one has $A_{2k-1} \geq B_{2k-1}$ for all $k \geq 1$. Further, $2kA_{2k} \leq A_{2k-1}$ for $k \geq 1$ and $2kB_{2k} \leq B_{2k-1}$ for $k \geq 1$.

Suppose now the sequence $\{\beta_n\}_{n\geq 0}$ belongs to $\mathcal{AMS}^*$. We first observe that the measure μ must be supported on $(0,2)$. Indeed, if $\mu\{[2,b]\} > 0$, then it is easy to see that the series $\sum_{k\geq 0} a_{2k} S_1(2k,1)$ is divergent; similarly $\mu(\{0\}) > 0$ leads to the divergence of the same series. Next, $\{\beta_n\}_{n\geq 0}$ satisfies condition (A) so that

$$\sum_{k\geq 0}\sum_{r\geq 0} a_k S_1(k,r) n^r < \infty$$

for all $n \geq 0$. Since $a_{2k} \geq A_{2k}$ for all $k \geq 0$, one has

$$\sum_{k\geq 0}\sum_{r\geq 0} A_{2k} S_1(2k,r) n^r < \infty$$

for all $n \geq 0$. Using the fact $(2k+1)A_{2k+1} \leq A_{2k}$ for $k \geq 0$ and the recurrence relation (R1), one also has $A_{2k+1}S_1(2k+1,r)n^r \leq A_{2k}S_1(2k,r)n^r + nA_{2k}S_1(2k,r-1)n^{r-1}$ for all $k \geq 0, n \geq 1$. This shows that $\sum_{k\geq 0}\sum_{r\geq 0} A_{2k+1}S_1(2k+1,r)n^r < \infty$ for all $n \geq 0$. Thus $\sum_{k\geq 0}\sum_{r\geq 0} A_k S_1(k,r)n^r < \infty$ for all $n \geq 0$. It then follows from

$$\sum_{k\geq 0}\sum_{r\geq 0} A_k S_1(k,r) n^r = \sum_{k\geq 0}\sum_{r\geq 0} \Big[\int_{[1,2)} \frac{(x-1)^k}{k!}\, d\mu(x)\Big] S_1(k,r) n^r$$

$$= \int_{[1,2)} \Big[\sum_{k\geq 0} \sum_{r \geq 0} \frac{(x-1)^k}{k!} S_1(k,r) n^r \Big] \, d\mu(x)$$

$$= \int_{[1,2)} \frac{1}{(2-x)^n} \, d\mu(x)$$

for all $n \geq 1$ that (A*) is satisfied. (We used the formula (G2) here). Further,

$$\begin{aligned} b_r &= \sum_{k\geq 0} (-1)^{k-r} a_k S_1(k,r) \\ &= \sum_{k \geq 0} (-1)^{k-r} \Big[\int_{(0,2)} \frac{(x-1)^k}{k!} \, d\mu(x) \Big] S_1(k,r) \\ &= \int_{(0,2)} \Big[(-1)^r \sum_{k\geq 0} \frac{(1-x)^k}{k!} S_1(k,r) \Big] \, d\mu(x) \\ &= \int_{(0,2)} \frac{(\log x)^r}{r!} \, d\mu(x) \end{aligned}$$

for all $r \geq 1$, so that (B*) holds. (We used the formula (G1) here.)

Conversely, suppose μ is supported on $(0,2)$ and (A*) and (B*) hold. Our observations above show that

$$\sum_{k\geq 0} \sum_{r\geq 0} A_k S_1(k,r) n^r < \infty$$

for all $n \geq 0$. Clearly, $a_{2k-1} \leq A_{2k-1}$ for all $k \geq 1$ so that, to verify condition (A), we need only verify the finiteness of

$$\sum_{k\geq 0} \sum_{r\geq 0} a_{2k} S_1(2k,r) n^r \qquad (n \geq 0).$$

Note that $a_{2k} = A_{2k} + B_{2k} \leq \frac{A_{2k-1}}{2k} + \frac{B_{2k-1}}{2k} \leq \frac{2A_{2k-1}}{2k}$ for all $k \geq 1$. Using the recurrence relation (R1), we have

$$\begin{aligned} a_{2k} S_1(2k,r) n^r &\leq \frac{2A_{2k-1}}{2k} S_1(2k,r) n^r \\ &\leq 2A_{2k-1} S_1(2k-1,r) n^r + n A_{2k-1} S_1(2k-1, r-1) n^{r-1} \end{aligned}$$

for all k and $n \geq 1$, and it is now clear that (A) holds. Further, (B*) yields the non-negativity of b_r $(r \geq 0)$ in view of our computations earlier so that (B) holds as well, and $\{\beta_n\}_{n\geq 0}$ falls in $\mathcal{AMS}^*$. $\square$

Examples 2. (i) As noted in (ii) of Examples 1 above, the sequence $\{\beta_n = (1+\delta)^n\}_{n\geq 0}$, with $0 \leq \delta < 1$, is a member of $\mathcal{A}MS^*$. The sequence $\{\beta_n\}_{n\geq 0}$ satisfies the hypotheses of Theorem 6 with the corresponding measure μ being the unit point mass at $(1+\delta)$. Clearly, the sequence $\{\beta_n\}_{n\geq 0}$ can be looked upon as $\{\theta_n(T)\}_{n\geq 0}$ corresponding to a weighted shift T that is an alternatingly hyperexpansive subnormal operator.
(ii) The function $f(x) = (a^x + a^{-x})/2$, where $1 < a < 2$, defines a member of $\mathcal{A}MF^*$ such that the sequence $f(n)$ satisfies the hypotheses of Theorem 6; the corresponding weighted shift is alternatingly hyperexpansive and subnormal with the associated measure μ concentrated at the points a and $1/a$. On the other hand, the absolutely monotone moment sequence $\{\cosh(n)\}_{n\geq 0}$ does not satisfy the hypotheses of Theorem 6.
(iii) In view of Theorems 2 and 3, the sequence $\{\beta_n = f'(n)\}_{n\geq 0}$ that is interpolated by the derivative $f'(x)$ of the function $f(x)$ in (ii) above is a member of $\mathcal{A}MS^*$; however, the weighted shift T with $\theta_n(T) = 1 + \beta_n$, though alternatingly hyperexpansive, is non-subnormal.

Motivated by our observations so far, we raise the following questions.

Questions. If $T = \Gamma(\alpha_n)$ is such that $\{\theta_n(T)\}_{n\geq 0}$ is interpolated by a member of $\mathcal{AMF}^*$, does the limit $\lim_{n\to\infty} \alpha_n$ exist? More generally, does the limit $\lim_{n\to\infty} \alpha_n$ exist if $\{\theta_n(T)\}_{n\geq 0}$ is interpolated by an absolutely monotone function on $\mathbb{R}_+$? In general, does the limit $\lim_{n\to\infty} \alpha_n$ exist for any alternatingly hyperexpansive weighted shift $T = \Gamma(\alpha_n)$?

REFERENCES

[1] A. ATHAVALE, Alternatingly hyperexpansive operator tuples, Positivity **5** (2001), 259–273.

[2] C. BERG, J. P. R. CHRISTENSEN AND P. RESSEL, Harmonic Analysis on Semigroups, Springer-Verlag, Berlin, 1984.

[3] J. B. CONWAY, The Theory of Subnormal Operator, Math. Surveys and Monographs, Vol. 36, Amer. Math. Soc., Providence, Rhode Island, 1991.

[4] M. R. EMBRY, A generalization of the Halmos–Bram criterion for subnormality, Acta Sci. Math. (Szeged) **31** (1973), 61–64.

[5] R. L. GRAHAM, D. E. KNUTH AND O. PATASHNIK, Concrete Mathematics, Addison-Wesley, New York, 1994.

[6] F. B. Hildebrandt, Introduction to Numerical Analysis, McGraw-Hill, New York, 1974.

[7] Zenon J. Jablonski, Complete hyperexpansivity, subnormality and inverted boundedness conditions, Int. Eq. Operator Th. **44** (2002), 316–336.

[8] W. Rudin, Real and Complex Analysis, McGraw-Hill, New York, 1986.

[9] V. M. Sholapurkar and A. Athavale, Completely and alternatingly hyperexpansive operators, J. Operator Theory **43** (2000), 43–68.

[10] A. Shields, Weighted shift operators and analytic function theory, Topics in Operator Theory, Math. Surveys **13**, Amer. Math. Soc., Providence, R.I. (1974), 49–128.

[11] D. Widder, The Laplace Transform, Princeton University Press, London, 1946.

Ameer Athavale,
Department of Mathematics,
Indian Institute of Technology Bombay,
Mumbai 40007,
India
athavale@math.iitb.ac.in

Abhijit Ranjekar,
Department of Mathematics,
University of Pune,
Pune 411007,
India
abhijit@math.unipune.ernet.in

V. M. Sholapurkar,
S. P. College,
Pune 411030,
India
v_sholapurkar@vsnl.net

Received on 24 December 2003.