Discrete Characterizations of Exponential Dichotomy for Evolution Families

PETRE PREDA, ALIN POGAN AND CIPRIAN PREDA

ABSTRACT. We present some characterizations of exponential dichotomy using a discrete argument. The results obtained generalize to the case of exponential dichotomy some theorems proved by Littman, Rolewicz and Zabczyk.

1. INTRODUCTION

One of the most remarkable results in the theory of stability for a strongly continuous semigroup of linear operators has been obtained by Datko [2] in 1970; it states that the semigroup $T = \{T(t)\}_{t\geq 0}$ is uniformly exponentially stable if and only if, for each vector x from the Banach space X, the application $t \mapsto \|T(t)x\|$ lies in $L^2(\mathbf{R}_+)$. Later, A. Pazy (see for instance [10]) showed that the result remains true even if we replace $L^2(\mathbf{R}_+)$ with $L^p(\mathbf{R}_+)$, where $p \in [1, \infty)$. In 1973, R. Datko [3] generalized the results above as follows.

Theorem 1.1. *An evolutionary process $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ with exponential growth is uniformly exponentially stable if and only if there is $p \in [1,\infty)$ such that*

$$\sup_{s\geq 0} \int_s^\infty \|U(t,s)x\|^p dt < \infty \qquad (x \in X).$$

The result provided by Theorem 1.1 was extended to dichotomy by P. Preda and M. Megan [14] in 1985. The same result was generalized in 1986 by S. Rolewicz [16] in the following way.

2000 *Mathematics Subject Classification. Primary* 34D09; *Secondary* 34D05, 39A12, 47D06.

Key words and phrases. Evolution families, exponential dichotomy.

Theorem 1.2. *Let $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a continuous, nondecreasing function with $\phi(0) = 0$ and $\phi(u) > 0$ for each positive u, and $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ an evolutionary process on X with exponential growth. If*

$$\sup_{s\geq 0} \int_s^\infty \phi\big(\|U(t,s)x\|\big)\, dt < \infty \qquad (x \in X),$$

then $\mathcal{U}$ is uniformly exponentially stable.

We note here the result obtained independently by Littman [6] in 1989, in the case of C_0-semigroups but without the assumption of continuity of ϕ.

Results of this type, for the case of C_0-semigroups were provided by I. Zabczyk [17] in 1974, with the additional requirement that the function ϕ is also convex, as can be seen below:

Theorem 1.3. *For every C_0-semigroup $T = \{T(t)\}_{t\geq 0}$ the following statements are equivalent:*

(i) *T is exponentially stable;*

(ii) *there is a convex increasing function $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ vanishing at 0 and for every $x \in X$ there is $\alpha(x) > 0$ such that*

$$\int_0^\infty \phi\big(\alpha(x)\|T(t)x\|\big)\, dt < \infty \qquad (x \in X);$$

(iii) *there is a convex increasing function $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ with $\varphi(0) = 0$ and for every $x \in X$ there is $\alpha(x) > 0$ such that*

$$\sum_{n=0}^{\infty} \varphi\big(\alpha(x)\|T(n)x\|\big) < \infty \qquad (x \in X).$$

Also, more recently, an unified treatment was presented by J. M. A. M. Neerven [8] in terms of Banach functions spaces.

The aim of this paper is to extend the preceding results to the case of exponential dichotomy using a discrete time argument.

2. PRELIMINARIES

In the beginning we will fix some standard notation. We denote by $\mathcal{A}$ the set of all non-decreasing functions $a : \mathbf{R}_+ \to \mathbf{R}_+$ with the property that $a(t) > 0$ for all $t > 0$. In what follows we will put X for a Banach space and $\mathcal{B}(X)$ the Banach algebra of all linear and bounded operators acting on X.

Remark 2.0. *If* $a \in \mathcal{A}$ *and* $A : \mathbf{R}_+ \to \mathbf{R}_+, A(u) = \int_0^u a(s)ds$, *then* $A \in \mathcal{A}$ *and* A *is a continuous convex bijection.*

Definition 2.1. *A family of bounded linear operators acting on* X *and denoted by* $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ *is called an* evolution family *if the following properties hold:*

e_1) $U(t,t) = I$ *(the identity operator on* X*), for all* $t \geq 0$;
e_2) $U(t,s) = U(t,r)\ U(r,s)$, *for all* $t \geq r \geq s \geq 0$;
e_3) *there exist* $M, w > 0$ *such that*

$$\|U(t,s)\| \leq Me^{w(t-s)} \quad , \quad \text{for all} \quad t \geq s \geq 0.$$

In order to deal with the dichotomy property we give the following:

Definition 2.2. *A function* $P : \mathbf{R}_+ \to \mathcal{B}(X)$ *is said to be a* dichotomy projection family *if*

p_1) $P^2(t) = P(t)$, *for all* $t \geq 0$;
p_2) $P(\cdot)x$ *is a bounded function, for all* $x \in X$.

We also denote by $Q(t) = I - P(t)$, $t \geq 0$.

Definition 2.3. *An evolution family* $\mathcal{U}$ *is said to be* uniformly exponentially dichotomic (u.e.d.) *if there exists* P *a dichotomy projection family and two constants* $N, \nu > 0$ *such that the following conditions hold:*

d_1) $P(t)U(t,s) = U(t,s)P(s)$ *for all* $t \geq s \geq 0$;
d_2) $U(t,s) : KerP(s) \to KerP(t)$ *is an isomorphism for all* $t \geq s \geq 0$;
d_3) $\|U(t,s)x\| \leq Ne^{-\nu(t-s)}\|x\|$, *for all* $t \geq s \geq 0$, *and all* $x \in ImP(s)$;
d_4) $\|U(t,s)x\| \geq \frac{1}{N}e^{\nu(t-s)}\|x\|$, *for all* $t \geq s \geq 0$, *and all* $x \in KerP(s)$.

In what follows we will consider an evolution family $\mathcal{U}$ *for which there is a dichotomy projection family* P *such that the properties* d_1) *and* d_2) *hold. In this case we will denote by*

$$U_1(t,s) = U(t,s)|\ ImP(s) \quad , \quad U_2(t,s) = U(t,s)|\ KerP(s).$$

Even if all the conditions e_1), e_2), e_3) and d_1), d_2) are satisfied, it does not follows that U_2^{-1} has exponential growth, as the following example shows.

Example 2.4. Let $X = \mathbf{R}$, $U(t,s) = e^{-(t^2-s^2)}$, $P(t) = 0$. Then $U_2^{-1}(t,s) = e^{t^2-s^2}$, for all $t \geq s \geq 0$ and hence U_2^{-1} does not have exponential growth.

Remark 2.5. *The evolution family $\mathcal{U}$ is u.e.d. if and only if there exist the constants $N_1, N_2, \nu_1, \nu_2 > 0$ such that, for all $t \geq s \geq 0$,*

$$\|U_1(t,s)\| \leq N_1 e^{-\nu_1(t-s)} \quad \textit{and} \quad \|U_2^{-1}(t,s)\| \leq N_2 e^{-\nu_2(t-s)}.$$

Lemma 2.6. *Let $g : \{(t,s) \in \mathbf{R}^2 : t \geq s \geq 0\} \to \mathbf{R}_+$. If g satisfy the conditions*

i) $g(t,s) \leq g(t,r)g(r,s)$, *for all* $t \geq r \geq s \geq 0$*;*

ii) $\sup\limits_{0 \leq t_0 \leq t \leq t_0+1} g(t,t_0) < \infty$*;*

iii) *there exists $h : \mathbf{N} \to \mathbf{R}_+$ with $\lim\limits_{n\to\infty} h(n) = 0$ such that*

$$g(m+n_0, n_0) \leq h(m) \qquad (m,\, n_0 \in \mathbf{N}),$$

then there exist $N, \nu > 0$ such that

$$g(t,t_0) \leq Ne^{-\nu(t-t_0)} \qquad (t \geq t_0 \geq 0).$$

Proof. Let $a = \sup\limits_{0 \leq t_0 \leq t \leq t_0+1} g(t,t_0)$, $m_0 = \min\{m \in \mathbf{N} : h(m) \leq \frac{1}{e}\}$.

Conditions i) and ii) imply that $\sup\limits_{0 \leq t_0 \leq t \leq t_0+m_0} g(t,t_0) \leq a^{m_0}$. Fix $t, t_0 \geq 0$ with $t \geq t_0 + 2m_0$, $m = \left[\dfrac{t}{m_0}\right]$, $n = \left[\dfrac{t_0}{m_0}\right]$, where [s] denotes the largest integer less or equal than $s \in \mathbf{R}$. It follows that

$$mm_0 \leq t < (m+1)m_0, \; nm_0 \leq t_0 < (n+1)m_0, \; m \geq n+2,$$

and

$$\begin{aligned} g(t,t_0) &\leq g(t,mm_0)g(mn_0,(n+1)n_0)g((n+1)n_0,t_0) \\ &\leq a^{m_0} \prod_{k=n+2}^{m} g(km_0,(k-1)m_0)a^{m_0} = a^{2m_0} \prod_{k=n+2}^{m} h(m_0) \\ &\leq a^{2m_0} \prod_{k=n+2}^{m} e^{-1} = a^{2m_0} e^{-(m-n-1)} \leq a^{2m_0} e^{2-\frac{t-t_0}{m_0}} \end{aligned}$$

If $t_0 \leq t \leq t_0 + 2m_0$, then it follows easily that

$$g(t,t_0) \leq a^{2m_0} \leq a^{2m_0} e^{2-\frac{t-t_0}{m_0}} \quad ,$$

and hence that

$$g(t,t_0) \leq Ne^{-\nu(t-t_0)} \quad , \quad \text{for all} \quad t \geq t_0 \geq 0 \quad , \quad \text{where}$$

$$N = e^2 a^{2m_0} \quad , \quad \nu = \frac{1}{m_0}. \qquad \square$$

Lemma 2.7. *If $a \in \mathcal{A}$, $\alpha : \mathbf{N}^2 \to \mathbf{R}_+$, $v > 0$ satisfy the following conditions:*

i) $\sup\{\alpha(n,m) : m,n \in \mathbf{N}, n \leq k\} < \infty$;

ii) *there exists $C > 0$ such that $\sum\limits_{j=0}^{n} a\Big(\frac{1}{e^{vj}}\alpha(n,m)\Big) \leq C$, for all $m,n \in \mathbf{N}$, then $\sup\limits_{m,n\in\mathbf{N}} \alpha(n,m) < \infty$.*

Proof. Assume towards a contradiction that $\sup\limits_{m,n\in\mathbf{N}} \alpha(m,n) = \infty$. Having in mind that

$$\lim_{p\to\infty} a(e^{vp}) = \infty,$$

we find

$$\sum_{p=0}^{\infty} a(e^{vp}) = \infty,$$

which implies that there exists $k_0 \in \mathbf{N}$ such that

$$\sum_{p=0}^{k_0} a(e^{vp}) \geq C + 1.$$

By our assumption and by condition i) it follows that

$$\sup_{m\geq 0, n\geq k_0} \alpha(n,m) = \infty,$$

and so there exist $m_0, n_0 \in \mathbf{N}$, with $n_0 \geq k_0$ and $\alpha_{m,n_0} \geq e^{wk_0}$. Now it is easy to check that

$$C \geq \sum_{j=0}^{n_0} a\Big(\frac{1}{e^{vj}}\alpha(n_0,m_0)\Big) \geq \sum_{j=0}^{n_0} a\Big(\frac{1}{e^{vj}}e^{vk_0}\Big)$$

$$\geq \sum_{j=0}^{k_0} a\Big(e^{v(k_0-j)}\Big) = \sum_{p=0}^{k_0} a(e^{vp}) \geq C + 1,$$

which is a contradiction. $\square$

3. The Main Result

We start with the following

Lemma 3.1. *If $a \in \mathcal{A}$ is a continuous convex function and if $T : \mathbf{N}^2 \to \mathcal{B}(X)$ is an operator-valued function with the property that*

$$\sup_{m\in\mathbf{N}} \sum_{n=0}^{\infty} a(\|T(m,n)x\|) < \infty \qquad (x \in X),$$

then there exist $j_0 \in \mathbf{N}, r_0 > 0$ such that

$$\sup_{m\in\mathbf{N}} \sum_{n=0}^{\infty} a(\|T(m,n)x\|) \leq j_0 \qquad (x \in X \quad \textit{with} \quad \|x\| \leq r_0).$$

Proof. For every natural number j we consider the set

$$H_j = \{x \in X : \sup_{m\in\mathbf{N}} \sum_{n=0}^{\infty} a(\|T(m,n)x\|) \leq j\}.$$

From the fact that a is continuous it follows that H_j is a closed set and since a is also convex it follows that H_j is a convex set for all $j \in \mathbf{N}$. Using the hypothesis we can state that

$$X = \bigcup_{j=0}^{\infty} H_j.$$

By Baire's theorem it follows that there exists $j_0 \in \mathbf{N}$ such that H_{j_0} has nonempty interior. Then there are $x_0 \in X$ and $r_0 > 0$ such that every $y \in X$ with $\|y - x_0\| \leq r_0$ belongs to H_{j_0}. Let $x \in X$ with $\|x\| \leq r_0$ and $x_1 = x + x_0,\ x_2 = x - x_0$. Then $\|x_1 - x_0\| = \| - x_2 - x_0\| = \|x\| \leq r_0$ and hence $x_1,\ -x_2,\ x_2 \in H_{j_0}$. Finally, by convexity of H_{j_0} we obtain that

$$x = \frac{1}{2}x_1 + \frac{1}{2}x_2 \in \frac{1}{2}H_{j_0} + \frac{1}{2}H_{j_0} = H_{j_0}. \qquad \square$$

Now, we can state the main result of this paper.

Theorem 3.2. *The evolution family $\mathcal{U}$ is u.e.d. if and only if there exist $a, b \in \mathcal{A}$ such that, for all $x \in X$,*

$$\sup_{m\in\mathbf{N}} \sum_{k=0}^{\infty} a(\|U_1(k+m,m)P(m)x\|) < \infty \quad \textit{and}$$
$$\sup_{m\in\mathbf{N}} \sum_{k=0}^{m} b(\|U_2^{-1}(m,k)Q(m)x\|) < \infty.$$

Proof. Necessity. It is a simple computation for $a(t) = b(t) = t$.

Sufficiency. **Step 1**. Let us define

$$\alpha : \mathbf{N}^2 \to \mathbf{R}_+, \alpha(n,m) = \frac{1}{M}\|U_1(n+m,m)Q(m)x\|$$

where $x \in X$ is fixed arbitrary. It follows that

$$\begin{aligned}\sum_{j=0}^{n} a\Big(\frac{1}{e^{wj}}\alpha(n,m)\Big) &= \sum_{k=0}^{n} a\Big(\frac{1}{Me^{w(n-k)}}\|U_1(n+m,m)P(m)x\|\Big)\\ &\le \sum_{k=0}^{n} a(\|U_1(k+m,m)P(m)x\|)\\ &\le \sup_{m\in\mathbf{N}}\sum_{k=0}^{\infty} a(\|U_1(k+m,m)P(m)x\|) < \infty,\end{aligned}$$

for all $m,n \in \mathbf{N}$. By Lemma 2.7, it follows that $\sup\limits_{m,n\in\mathbf{N}} \alpha(n,m) < \infty$, and hence by the principle of uniform boundedness we obtain that there exists $L_1 > 0$ such that, for all $m,n \in \mathbf{N}$,

$$\|U_1(n+m,m)P(m)\| \le L_1.$$

Now it is easy to see that

$$\begin{aligned}&\sup_{m\in\mathbf{N}}\sum_{k=0}^{\infty} A(\|U_1(k+m,m)P(m)x\|)\\ &\qquad\le L_1\|x\| \sup_{m\in\mathbf{N}}\sum_{k=0}^{\infty} a(\|U_1(k+m,m)P(m)x\|) < \infty,\end{aligned}$$

for all $x \in X$, where A is the function defined in Remark 2.0, which belongs to $\mathcal{A}$ and is continuous and convex, and hence, by applying Lemma 3.1 to the operator-valued function $T : \mathbf{N}^2 \to \mathcal{B}(X)$, $T(m,k) = U_1(k+m,m)P(m)$ it results that there exist $j_1 \in \mathbf{N}$ and $r_1 > 0$ such that

$$\sum_{k=0}^{\infty} A(\|U_1(k+m,m)P(m)x\|) \le j_1$$

for all $m \in \mathbf{N}$ and all $x \in X$ with $\|x\| \leq r_1$. A simple computation shows that

$$\begin{aligned}
&\sum_{k=0}^{n} A(\|U_1(n+m,m)P(m)x\|) = \\
&= \sum_{k=0}^{n} A(\|U_1(n+m,m+k)P(m+k)U_1(m+k,m)P(m)x\|) \\
&\leq \sum_{k=0}^{n} A(L_1\|U_1(m+k,m)P(m)x\|) \\
&= \sum_{k=0}^{n} A(\|U_1(m+k,m)P(m)(L_1x)\|) \leq j
\end{aligned}$$

for all $m, n \in \mathbf{N}$, and each $x \in X$ with $\|x\| \leq \dfrac{r_1}{L_1}$. Because A is also bijective we have that

$$\|U_1(n+m,m)\| \leq \frac{L_1}{r_1} A^{-1}\Big(\frac{j_1}{n+1}\Big) \qquad (m, n \in \mathbf{N}).$$

From Lemma 2.6 it follows that there exist the constants $N_1, \nu_1 > 0$ such that

$$\|U_1(t,s)\| \leq N_1 e^{-\nu_1(t-s)} \qquad (t \geq s \geq 0).$$

Step 2. Now fix $x \in X$ arbitrary and consider $\beta : \mathbf{N}^2 \to \mathbf{R}_+$,

$$\beta(n,m) = \frac{1}{M}\|U_2^{-1}(n+1,n)Q(n+1)x\|.$$

Then we have

$$\begin{aligned}
\sum_{j=0}^{n} b\Big(\frac{1}{e^{wj}}\beta(n,m)\Big) &= \sum_{k=0}^{n} b\Big(\frac{1}{Me^{w(n-k)}}\|U_2^{-1}(n+1,n)Q(n+1)x\|\Big) \\
&= \sum_{k=0}^{n} b\Big(\frac{1}{Me^{w(n-k)}}\|U_2(n,k)U_2^{-1}(n+1,k)Q(n+1)x\|\Big)
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{k=0}^{n} b(\|U_2^{-1}(n+1,k)Q(n+1)x\|)\\
&\le \sum_{k=0}^{n+1} b(\|U_2^{-1}(n+1,k)Q(n+1)x\|)\\
&\le \sup_{l\in\mathbf{N}} \sum_{k=0}^{l} b(\|U_2^{-1}(l,k)Q(l)x\|) < \infty,
\end{aligned}
$$

for all $n, m \in \mathbf{N}$. As a consequence of Lemma 2.7. we obtain that

$$\sup_{n\in\mathbf{N}} \|U_2^{-1}(n+1,n)Q(n+1)x\| < \infty \qquad (x \in X),$$

and by the principle of uniform boundedness it follows that

$$\sup_{n\in\mathbf{N}} \|U_2^{-1}(n+1,n)\| < \infty.$$

Now it is clear that there exists a constant $\delta > 0$ such that

$$\|U_2^{-1}(n,m)\| \le e^{\delta(n-m)} \qquad (n \ge m).$$

For x an arbitrary vector of X, we define

$$\gamma : \mathbf{N}^2 \to \mathbf{R}_+\ ,\ \gamma(n,m) = \|U_2^{-1}(n+m,m)Q(n+m)x\|.$$

We have that, for all $m, n \in \mathbf{N}$,

$$
\begin{aligned}
&\sum_{j=0}^{n} b\Big(\frac{1}{e^{\delta j}}\gamma(n,m)\Big)\\
&\quad = \sum_{j=0}^{n} b\Big(\frac{1}{e^{\delta j}}\|U_2^{-1}(j+m,m)U_2^{-1}(n+m,j+m)Q(m+n)x\|\Big)\\
&\quad \le \sum_{j=0}^{n} b(\|U_2^{-1}(n+m,j+m)Q(m+n)x\|)\\
&\quad \le \sum_{k=0}^{n+m} b(\|U_2^{-1}(n+m,k)Q(n+m)x\|)\\
&\quad \le \sup_{l\in\mathbf{N}} \sum_{k=0}^{l} b(\|U_2^{-1}(l,k)Q(l)x\|) < \infty.
\end{aligned}
$$

By applying once again Lemma 2.7 we have that $\sup\limits_{n,m\in\mathbf{N}} \gamma(n,m) < \infty$, and hence by the principle of uniform boundedness we obtain that there exists $L_2 > 0$ such that, for all $n, m \in \mathbf{N}$,

$$\|U_2^{-1}(n+m,m)Q(n+m)\| \leq L_2.$$

Then it is easy to observe that

$$\sup_{m\in\mathbf{N}} \sum_{k=0}^{m} B(\|U_2^{-1}(m,k)Q(m)x\|) \\ \leq L_2\|x\| \sup_{m\in\mathbf{N}} \sum_{k=0}^{m} b(\|U_2^{-1}(m,k)Q(m)x\|) < \infty$$

for all $x \in X$, where $B : \mathbf{R}_+ \to \mathbf{R}_+$, $B(u) = \int\limits_0^u b(s)ds$, which, by Remark 2.0, is continuous and convex. If we apply Lemma 3.1 to the operator-valued function $V : \mathbf{N}^2 \to B(X)$ defined by

$$V(n,m) = \begin{cases} U_2^{-1}(m,n)Q(m) & , \quad m \geq n \\ 0 & , \quad m < n \end{cases}$$

we can state that there are $j_2 \in \mathbf{N}, r_2 > 0$ such that

$$\sup_{m\in\mathbf{N}} \sum_{k=0}^{m} B(\|U_2^{-1}(m,k)Q(m)x\|) \leq j_2,$$

for all $x \in X$ with $\|x\| \leq r_2$. It follows that

$$\begin{aligned} & \sum_{k=0}^{n} B(\|U_2^{-1}(m+n,m)Q(m+n)x\|) \\ = & \sum_{k=0}^{n} B(\|U_2^{-1}(k+m,m)Q(k+m) \times \\ & \qquad \times U_2^{-1}(m+n,m+k)Q(m+n)x\|) \\ \leq & \sum_{k=0}^{n} B(L_2\|U_2^{-1}(m+n,k+m)Q(m+n)x\|) \\ = & \sum_{j=m}^{m+n} B(\|U_2^{-1}(n+m,j)Q(n+m)(L_2x)\|) \\ \leq & \sum_{j=0}^{n+m} B(\|U_2^{-1}(m+n,j)Q(m+n)(L_2x)\|) \leq j_2, \end{aligned}$$

for all $m,n \in \mathbf{N}$, and all $x \in X$ with, $\|x\| \leq \frac{r_2}{L_2}$. Using the fact that B is bijective too we obtain that, for all $m,n \in \mathbf{N}$,

$$\|U_2^{-1}(n+m,m)\| \leq \frac{L_2}{r_2} B^{-1}\Big(\frac{j_2}{n+1}\Big).$$

In order to apply Lemma 2.6 we observe that

$$U_2^{-1}(t,t_0) = U_2(t_0,[t_0])\, U_2^{-1}([t_0]+2,[t_0])\, U_2([t_0]+2,t)$$

for all $0 \leq t_0 \leq t \leq t_0+1$ and hence

$$\sup_{0\leq t_0\leq t\leq t_0+1} \|U_2^{-1}(t,t_0)\| \leq M^2 L_2 e^{3\omega}.$$

Finally we obtain that there exists $N_2, \nu_2 > 0$ such that

$$\|U_2^{-1}(t,t_0)\| \leq N_2 e^{-\nu_2(t-t_0)} \quad , \quad \text{for all} \quad t \geq t_0 \geq 0. \qquad \square$$

The necessity of Theorem 3.2 is not true for all $a,b \in \mathcal{A}$ as the following example illustrates.

Example 3.3. Let $X = \mathbf{R}$, $U(t,s) = e^{-(t-s)}$, $P(t) = 1, a(u) = \sum\limits_{n=1}^{\infty} \frac{\sqrt[n]{u}}{n^2}$. It is clear that $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ is u.e.d. but for $x=1$, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} a(\|U_1(k+m,m)P(m)x\|) &= \sum_{k=0}^{\infty} a(e^{-k}) = \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n^2} e^{-\frac{k}{n}} \\
&= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{1}{n^2} e^{-\frac{k}{n}} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{e^{\frac{1}{n}}}{e^{\frac{1}{n}}-1} = \infty,
\end{aligned}$$

for all $m \in \mathbf{N}$.

Theorem 3.4. *The evolution family $\mathcal{U}$ is u.e.d. if and only if there exist $K, L, p, q > 0$ such that*

$$\sum_{n=m}^{\infty} \|U(n,m)x\|^p \leq K\|x\|^p \qquad (m \in \mathbf{N},\ x \in ImP(m)) \quad \textit{and}$$

$$\sum_{n=m}^{l} \|U(n,m)x\|^q \leq L\|U(l,m)x\|^q \qquad (m \geq l,\ x \in KerP(m)).$$

Proof. Follows easily from Theorem 3.2 for $a(u) = u^p, b(u) = u^q$. $\square$

REFERENCES

[1] C. Chicone, Y. Latushkin, *Evolution semigroups in dynamical systems and differential equations*, Mathematical Surveys and Monographs **70**, American Mathematical Soxiety, Providence, RI, 1999.
[2] R. Datko, *Extending a theorem of Liapunov to Hilbert spaces*, J. Math. Anal. Appl. **32** (1970), 610 – 616.
[3] R. Datko, *Uniform asymptotic stability of evolutionary processes in a Banach space*, SIAM J. Math. Analysis **3** (1973), 428 – 445.
[4] D. Henry, *Geometric theory of semi-linear parabolic equations*, Springer-Verlag, New York, 1981.
[5] J. P. La Salle, *The stability and control of discrete processes*, Springer-Verlag, Berlin, 1990.
[6] W. Littman, *A generalization of the theorem Datko–Pazy*, Lecture Notes in Control and Inform. Sci. **130** (1983), 318 – 323.
[7] M. Megan, A. Pogan, *On a theorem of Rolewicz for semigroups of operators in locally convex spaces*, Ann. Math. Blaise Pascal **7** (2000), 23 – 35.
[8] J. M. A. M. van Neerven, *Exponential stability of operators and semigroups*, J. Func. Anal. **130** (1995), 293 – 309.
[9] J. M. A. M. van Neerven, *The asymptotic behavior of semigroups of linear operators*, Theory, Advances and Applications **88**, Birkhäuser, 1996.
[10] A. Pazy, *On the applicability of Liapunov's theorem in Hilbert spaces*, SIAM J. Math. Anal. Appl. **3** (1972), 291 – 294.
[12] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, 1983.
[13] M. Pinto, *Discrete dichotomies*, Computers Math. Appl. **28** (1994), 259 – 270.
[14] P. Preda, M. Megan, *Exponential dichotomy of evolutionary processes in Banach spaces*, Czech. Math. J. **35** (1985), 312 – 323.
[15] K. M. Przyluski, S. Rolewicz, *On stability of linear time-varying infinite-dimensional discrete-time systems*, Systems Control Lett. **4** (1994), 307 – 315.
[16] S. Rolewicz, *On uniform N-equastabily*, J. Math. Anal. Appl. **115** (1986), 434 – 441.
[17] J. Zabczyk, *Remarks on the control of discrete-time distributed parameter systems*, SIAM J. Control. Optim. **12** (1974), 721 – 735.
[18] Q. Zheng, *The exponential stability and the perturbation problem of linear evolution systems in Banach spaces*, J. Sichuan Univ. **25** (1988), 401 – 411.

Department of Mathematics,
West University of Timişoara,
Bd. V. Pârvan, no. 4,
1900–Timişoara, Romania
preda@math.uvt.ro, ciprian.preda@fse.uvt.ro

Received on 1 April 2003 and in revised form on 14 January 2004.