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Lipschitz Character of Solutions to the Double Inner Obstacle Problems

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ABSTRACT. In our paper we analyse the double inner problem with one impediment from below and one impediment from above. Assuming the Lipschitz character of the obstacles we show that the corresponding solution is also Lipschitz. We extend here the result obtained by Stampacchia and Vignoli (1972) who considered the inner obstacle problem with a single impediment. Our work is based on the ideas introduced by J. Jordanov in 1982.

1. Introduction

The study of the Lipschitz character of the solutions to the obstacle problems was initiated in [6]. The authors proved that the solution of the global problem and of the inner problem with a single impediment is Lipschitz continuous assuming that the impediments are Lipschitz. Later on the paper [2] was published, where one can find theorems giving the Lipschitz continuity of the solutions to the global inverse and double global problems.

We aim at transferring results concerning Lipschitz character of the solutions of the global obstacle problems to the case of the inner ones. In order to do so we construct appropriate extensions of the inner impediments to the global ones which enable us to identify the inner problem with the corresponding global one.

The present paper is a part of the research program on free boundary problems.

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2. Notation and Basic Definitions

Throughout the paper we assume that $\Omega \subset \mathbb{R}^n$ is an open, bounded set with smooth boundary $\partial\Omega$. The functions $a_{ij} : \bar{\Omega} \to \mathbb{R}$ for $1 \le i, j \le n$ belong to $C^1(\Omega)$ and satisfy the ellipticity condition, i.e. there exist $\gamma, \mu > 0$ such that

$$\mu|\xi|^2 \ge a_{ij}(x)\xi_i\xi_j \ge \gamma|\xi|^2 \quad \text{for } x \in \Omega \text{ and } \xi \in \mathbb{R}^n, \tag{1}$$

where the summation convention is adopted. We also introduce the second order elliptic operator

$$L = -\partial_{x_i}\big(a_{ij}(x)\partial_{x_j}\big). \tag{2}$$

Remark 2.1. The operator L defined by (2) considered as the mapping $L : H_0^1(\Omega) \to H^{-1}(\Omega)$ defines a bilinear, continuous and coercive form on $H_0^1(\Omega)$ as follows (see [5]):

$$a(u, v) = \langle Lu, v\rangle = \int_\Omega a_{ij}(x)u_{x_i}(x)v_{x_j}(x)\,dx \qquad (u, v \in H_0^1(\Omega)). \tag{3}$$

Now we pass to the precise definitions of fundamental concepts of this work.

Let us consider a function $\Psi_1 \in H^1(E)$, where E is a compact set such that $E \subset \Omega$ and ∂E is smooth. Next we take a function $\Phi_1 \in H^1(F)$, where F is a compact set such that $F \subset \Omega$ and ∂F is smooth. Moreover, we assume that:

$$\Psi_1 \le \Phi_1 \quad \text{on } E \cap F. \tag{4}$$

We denote by K_1^1 the following admissible set:

$$K_1^1 = \{v \in H_0^1(\Omega) \ : \ v \ge \Psi_1 \text{ on } E \ \wedge v \le \Phi_1 \text{ on } F\}. \tag{5}$$

Definition 2.2. *For the form defined by (3) and $f \in H^{-1}(\Omega)$ the problem:*

Find $u_1^1 \in K_1^1$ such that

$$a(u_1^1, v - u_1^1) \ge \langle f, v - u_1^1\rangle \quad \text{for any } v \in K_1^1, \tag{6}$$

where K_1^1 is defined by (5) is called a double inner obstacle problem *with the impediments Ψ_1 and Φ_1.*

In what follows we shall use the notation *DIP* to denote the double inner obstacle problem.

Remark 2.3. If we take $E = F = \Omega$ and assume that $\Psi_1 = \Psi$ and $\Phi_1 = \Phi$ are such that $\Psi\big|_{\partial\Omega} \le 0$, $\Phi\big|_{\partial\Omega} \ge 0$ and $\Psi \ge \Phi$ then Definition 2.2 gives the definition of the double global obstacle problem with the impediments Ψ and Φ. The admissible set will be denoted by $\tilde{K}_1^1$. We shall use the notation DGP to denote the *double global obstacle problem.*

The existence and uniqueness theorem for DGP can be found in [5].

3. Lipschitz Regularity

In this section we present the main result of our paper, i.e., the Lipschitz regularity of the solution of DIP. We start with recalling the following lemma (see [6], [2]).

Lemma 3.1. *The solution $\tilde{u}_1^1$ of DGP with the impediments Ψ and Φ is Lipschitz ($\tilde{u}_1^1 \in H^{1,\infty}(\Omega)$) provided Ψ, $\Phi \in H^{1,\infty}(\Omega)$.*

In the proof of the following theorem our ideas are based on equivalence of the inner problem with the corresponding global one.

Theorem 3.2. *If the functions $\Psi_1 \in H^{1,\infty}(E)$, $\Phi_1 \in H^{1,\infty}(F)$ satisfy*

$$\Psi_1 \leq \Phi_1 \quad \text{in } E \cap F, \tag{7}$$

then there exists a unique solution u_1^1 to DIP with the impediments Ψ_1, Φ_1 and f being equal to zero. Moreover, this solution is Lipschitz continuous.

Proof. Let us construct the function $\bar{\Psi}_1 : \Omega \to \mathbb{R}$ in the following way

$$\bar{\Psi}_1 = \begin{cases} w_1 & \text{in } \Omega \setminus E \\ \Psi_1 & \text{in } E, \end{cases} \tag{8}$$

where $w_1 \in H^1(\Omega \setminus E)$ solves the problem

$$\begin{cases} Lw_1 = 0 & \text{in } \Omega \setminus E \\ w_1 = \Psi_1 & \text{in } \partial E \\ w_1 = 0 & \text{in } \partial\Omega. \end{cases} \tag{9}$$

Next in a similar way we construct the function $\bar{\Phi}_1 : \Omega \to \mathbb{R}$.

$$\bar{\Phi}_1 = \begin{cases} w_2 & \text{in } \Omega \setminus F \\ \Phi_1 & \text{in } F, \end{cases} \tag{10}$$

where $w_2 \in H^1(\Omega \setminus F)$ solves the problem

$$\begin{cases} Lw_2 = 0 & \text{in } \Omega \setminus F \\ w_2 = \Phi_1 & \text{in } \partial F \\ w_2 = 0 & \text{in } \partial\Omega. \end{cases} \tag{11}$$

It is well known (see [5]) that in order to show existence and uniqueness of the solution u_1^1 of *DIP* it is enough to show non-emptiness of the set K_1^1. Let us note that $\max\{\bar{\Psi}_1, 0\} + \min\{\bar{\Phi}_1, 0\} \in K_1^1$ which yields the desired conclusion.

Now we proceed to regularity of the solution. We shall construct two Lipschitz functions Ψ and Φ such that the solution $\tilde{u}_1^1 \in \tilde{K}_1^1$ of *DGP* with the impediments Ψ and Φ and the force f being equal to zero will coincide with the solution u_1^1 of *DIP*.

Let us consider the coincidence set $I[u_1^1]$ for *DIP*. Obviously it is contained in $E \cup F$ (as (7) holds). We denote by $I_F[u_1^1]$ that part of $I[u_1^1]$ which is contained in $F \cap \overline{(\Omega \setminus E)}$ where $u_1^1 = \Phi_1$ and by $I_E[u_1^1]$ that part of $I[u_1^1]$ which is contained in $E \cap \overline{(\Omega \setminus F)}$ where $u_1^1 = \Psi_1$. Now we put

$$\tilde{\Psi}_1 = \begin{cases} \bar{\Psi}_1 & \text{in } E \\ \min\{\bar{\Psi}_1, \bar{\Phi}_1\} & \text{in } \Omega \setminus E, \end{cases} \tag{12}$$

$$\tilde{\Phi}_1 = \begin{cases} \bar{\Phi}_1 & \text{in } F \\ \max\{\bar{\Psi}_1, \bar{\Phi}_1\} & \text{in } \Omega \setminus F. \end{cases} \tag{13}$$

Both functions $\tilde{\Psi}_1$ and $\tilde{\Phi}_1$ are continuous. Moreover, they are both Lipschitz.

Now let us take a Lipschitz function $\xi \in H_0^{1,\infty}(\Omega)$ such that $\xi|_{\partial(\Omega \setminus E)} = 0$ and $\xi < 0$ in $\Omega \setminus E$. Next we consider the Lipschitz function $\delta \in H^{1,\infty}(\Omega)$ where we put $\delta = \tilde{\Psi}_1 + \xi$. We know that

$$u_1^1 = \Phi_1 = \tilde{\Phi}_1 \quad \text{in } \mathrm{I_F}[\mathrm{u_1^1}]. \tag{14}$$

It also satisfies

$$u_1^1 > \delta = \tilde{\Psi}_1 + \xi \quad \text{in } \mathrm{I_F}[\mathrm{u_1^1}] \tag{15}$$

since $\tilde{\Psi}_1 + \xi = \min\{\bar{\Psi}_1, \bar{\Phi}_1\} + \xi < \tilde{\Phi}_1$ in $I_F[u_1^1]$. From the continuity of u_1^1, ξ and $\tilde{\Psi}_1$ we state that there exists a neighbourhood O_F of $I_F[u_1^1]$ where the inequality (15) holds.

Now we choose a set D_F with the smooth boundary in the following way:

$$I_F[u_1^1] \subset D_F \subset \bar{D}_F \subset O_F \cap \overline{(\Omega \setminus E)}.$$

Let ψ be the solution of the problem:

$$\begin{cases} L\psi = 0 & \text{in } (\Omega \setminus E) \setminus \bar{D}_F \\ \psi = \delta & \text{in } \partial((\Omega \setminus E) \setminus \bar{D}_F). \end{cases} \tag{16}$$

The function ψ is Lipschitz. Next we remark that the set D was chosen in such a way that $(\Omega \setminus E) \setminus \bar{D}_F \subset \Omega \setminus I[u_1^1]$. Therefore using again the basic properties of the solutions to the obstacle problems (see [5]) we have that $Lu_1^1 = 0$ in $(\Omega \setminus E) \setminus \bar{D}_F$. Hence

$$L(u_1^1 - \psi) = 0 \quad \text{in } (\Omega \setminus E) \setminus \bar{D}_F.$$

Moreover, we have

$$u_1^1 = \psi = \bar{\Psi}_1 \quad \text{in } \partial\Omega,$$

it follows from definition of $\bar{\Psi}_1$ and the constructions of $\tilde{\Psi}_1$ and ξ,

$$u_1^1 \geq \psi = \bar{\Psi}_1 \quad \text{in } \partial E,$$

it follows from (7) and the constructions of $\tilde{\Psi}_1$ and ξ,

$$u_1^1 \geq \psi \quad \text{in } \partial D_F,$$

it follows from (15) and the construction of the set D_F. Then the maximum principle implies that

$$u_1^1 \geq \psi \quad \text{in } (\Omega \setminus E) \setminus \bar{D}_F. \tag{17}$$

Finally we put

$$\Psi = \begin{cases} \Psi_1 & \text{in } E \\ \delta & \text{in } \bar{D}_F \\ \psi & \text{in } (\Omega \setminus E) \setminus \bar{D}_F. \end{cases} \tag{18}$$

Clearly the function Ψ is Lipschitz continuous in Ω. Moreover,

$$u_1^1 \geq \Psi \quad \text{in } \Omega \tag{19}$$

since $u_1^1 \in K_1^1$, (17) holds and (15) is satisfied in $D_F \subset O_F \cap \overline{(\Omega \setminus E)}$.

Now we pass to the remaining part of the proof. We choose a Lipschitz function $\eta \in H_0^{1,\infty}(\Omega)$ such that $\eta > 0$ in $\Omega \setminus F$ and $\eta|_{\partial(\Omega \setminus F)} = 0$. Next we consider the Lipschitz function $\sigma \in H^{1,\infty}(\Omega)$ where we put $\sigma = \tilde{\Phi}_1 + \eta$. We know that

$$u_1^1 = \tilde{\Psi}_1 = \Psi_1 \quad \text{in } I_E[u_1^1]. \tag{20}$$

It also satisfies the following:

$$u_1^1 < \sigma = \tilde{\Phi}_1 + \eta \quad \text{in } I_E[u_1^1] \tag{21}$$

as $\tilde{\Psi}_1 = \Psi_1 < \tilde{\Phi}_1 + \eta$ in $I_E[u_1^1]$. From the continuity of u_1^1, η and $\tilde{\Phi}_1$ we state that there exists a neighbourhood O_E of $I_E[u_1^1]$ where the inequality (21) holds. Acting similarly as above we can choose a set D_E with the smooth boundary such that:

$$I_E[u_1^1] \subset D_E \subset \bar{D}_E \subset O_E \cap (\Omega \setminus F).$$

Denoting by ϕ the solution of the problem

$$\begin{cases} L\phi = 0 & \text{in } (\Omega \setminus F) \setminus \bar{D}_E \\ \phi = \sigma & \text{in } \partial((\Omega \setminus F) \setminus \bar{D}_E), \end{cases} \tag{22}$$

we get that ϕ is Lipschitz provided σ is Lipschitz. Knowing that $(\Omega \setminus F) \setminus \bar{D}_E \subset \Omega \setminus I[u_1^1]$ and using the maximum principle we deduce that:

$$u_1^1 \leq \phi \quad \text{in } (\Omega \setminus F) \setminus \bar{D}_E. \tag{23}$$

Finally we put

$$\Phi = \begin{cases} \Phi_1 & \text{in } F \\ \sigma & \text{in } \bar{D}_E \\ \phi & \text{in } (\Omega \setminus F) \setminus \bar{D}_E. \end{cases} \tag{24}$$

The function Φ is Lipschitz. Moreover,

$$u_1^1 \leq \Phi \quad \text{in } \Omega \tag{25}$$

as $u_1^1 \in K_1^1$, (23) holds and (21) is satisfied in $D_E \subset O_E \cap \overline{\Omega \setminus F}$.

Let us now denote by $\tilde{u}_1^1$ the solution of DGP with the impediments Ψ, Φ given by (18), (24), respectively and the force f being equal to zero. Conditions (19), (25) together with $u_1^1 \in H_0^1(\Omega)$ imply that $u_1^1 \in \tilde{K}_1^1$. Thus we can state

$$a(\tilde{u}_1^1, u_1^1 - \tilde{u}_1^1) \geq 0.$$

It is also true that

$$a(u_1^1, \tilde{u}_1^1 - u_1^1) \geq 0$$

because $u_1^1 \in K_1^1$ solves the variationally inequality (6) with $f = 0$ and $\tilde{u}_1^1 \in \tilde{K}_1^1 \subset K_1^1$. After adding the last two inequalities we shall obtain (using coercivity of the form $a(\cdot,\cdot)$ in $H_0^1(\Omega)$) that there exists $\nu > 0$ such that

$$\nu \parallel \tilde{u}_1^1 - u_1^1 \parallel^2 \leq a(\tilde{u}_1^1 - u_1^1, \tilde{u}_1^1 - u_1^1) \leq 0,$$

which implies that $\tilde{u}_1^1 = u_1^1$ in Ω. The Lipschitz continuity of $\tilde{u}_1^1$ (see Lemma 3.1) completes the proof. □

Remark 3.3. In this work we examine the Lipschitz regularity of the solution of the double inner problem. Its natural generalisation is the inner obstacle problem with $l > 1$ impediments from below and $m > 1$ from above. The main problem is to find appropriate extensions of the inner obstacles, which allow us to identify the solution of the inner problem with the solution of the corresponding global one.

Remark 3.4. It is well known that in case of solutions of the global problems one can expect their regularity up to $H^{2,p}$. For the inner problem the situation is much more complicated. Despite $H^{2,p}$ regularity of the obstacle the same class of the solution can not be obtained. However under certain assumptions it is possible to get $H^{2,p}$ regularity of the solutions (see [1], [3]).

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