Triangle Geometry and Jacobsthal Numbers

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Abstract. The convergence properties of certain triangle centres on the Euler line of an arbitrary triangle are studied. Properties of the Jacobsthal numbers, which appear in this process, are examined, and a new formula is given. A Jacobsthal decomposition of Pascal's triangle is presented.

This review article takes as its motivation a simple problem in elementary triangle geometry to study some properties of the Jacobsthal numbers, defined by the recurrence relation

$$a_{n+2} = a_{n+1} + 2a_n, a_0 = 0, a_1 = 1 \tag{1}$$

These numbers form the sequence $0, 1, 1, 3, 5, 11, 21, 43, \ldots$ [Sloane, A001045]. We let $J(n)$ or J_n stand for the nth Jacobsthal number, starting with $J(0)$=0. These numbers are linked to the binomial coefficients in a number of ways. Traditional formulas for $J(n)$ include

$$J(n) = \frac{1}{3.2^{n-1}} \sum_{k=1}^{floor((n+1)/2)} C(n, 2k-1)3^{2k-1} \tag{2}$$

$$J(n) = \sum_{j=0}^{floor(n/2)} C(n-1-j, j)2^j \tag{3}$$

To simplify expressions, we shall not normally give upper summation bounds in what follows, using the fact that $C(n, k) = 0$ for $k > n$ to ensure that all summations are finite.

The investigation of this article leads to another formula, namely

$$J(n) = \sum_{(n+k) \mod 3=1} C(n, k) = \sum_{(n+k) \mod 3=2} C(n, k) \tag{4}$$

which emphasizes how the Jacobsthal numbers provide an interesting decomposition property of Pascal's triangle. We shall also draw some links to the Fibonacci numbers.

The solution to recurrence (1) takes the form

$$J(n) = \frac{2^n}{3} - \frac{(-1)^n}{3}. \tag{5}$$

As may be expected of a direct generalization of the Fibonacci numbers (solutions of the recurrence $a(n) = a(n-1) + a(n-2), a(0) = 0, a(1) = 1$), the Jacobsthal numbers have many interesting properties.

The starting point of this study is a simple geometric exploration, concerning triangle centres along the Euler line of a plane triangle. We recall that a triangle centre is a point of concurrency of lines related to the geometry of the triangle. When these lines are drawn from the vertices of a triangle to the opposite sides they are commonly called cevians. An online database of significant triangle centres is maintained at

`[http://faculty.evansville.edu/ck6/encyclopedia/]`.

Examples are the centroid G of a triangle, obtained by joining vertices to the midpoints of the opposite sides, the circumcentre O, the point of intersection of the perpendicular bisectors through the midpoints of the sides, and the orthocentre H, the point of intersection of the altitudes of the triangle (the lines drawn from the vertices that meet the opposite sides at a right-angle). The points O, G, and H lie on a line called the Euler line. A further point that lies on this line is the point N, the centre of the nine-point circle. This circle is the circumcircle of the median triangle, which is obtained by joining the midpoints of the sides of the original triangle. These centres obey the relations

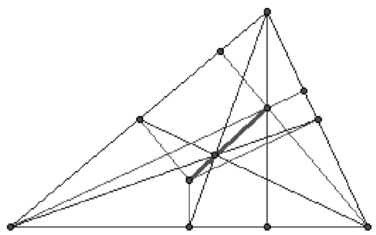
$$OH = 3OG = 2ON \tag{6}$$

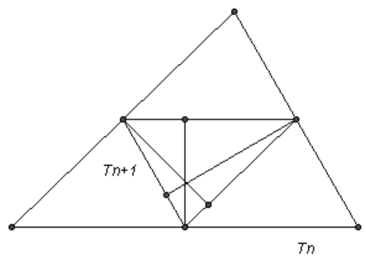
Note that for the purposes of this review, we shall refer to the line segment from O to H as the Euler line.

We now consider the sequence T_n of median triangles, associated with a given triangle T, defined as follows. We define $T_0 = T$, and T_{n+1} is defined to be the median triangle of T_n, obtained by joining the midpoints of the sides of T_n.

Some easy observations can be made. All triangles $T_n, n \geq 1$, are similar to T_0. In fact, it can be shown that $T_n = (-\frac{1}{2})^n T_0$.

By construction, the perpendicular bisectors of the sides of T_n are the altitudes of T_{n+1}, and hence the circumcentre O_n of T_n is the





orthocentre H_{n+1} of T_{n+1}:

$$O_n = H_{n+1} \tag{7}$$

We now use the properties of N, the nine-point circle centre, to generate another important relationship. Starting with the original triangle $T = T_0$, we note that $N = N_0$ is the circumcentre O_1 of the first median triangle T_1. This is by construction, since N is the centre of the unique circle which passes through the midpoints of the sides of T. We note in passing that it is also the circumcentre of the first orthic triangle (the triangle obtained by joining the feet of the altitudes).

Now it is well-known that $N_0 = N = \frac{1}{2}(O_0 + H_0)$ — that is, it is in the middle of the Euler line. Hence we have the relation

$$O_1 = \frac{1}{2}(O_0 + H_0). \tag{8}$$

This result can be generalized to the following.

Lemma 1.

$$O_{n+1} = \frac{1}{2}(O_n + H_n) \tag{9}$$

Proof. As above, O_{n+1} is the circumcentre of the median triangle T_{n+1} of T_n, so it is the nine-point circle centre N_n of T_n. The previous result applied to these two triangles yields the result. □

We now have the following proposition, which will link this construction to the Jacobsthal numbers.

Proposition 2.

$$O_n = a_n O_0 + b_n H_0 \tag{10}$$

where the sequences a_n and b_n obey the recurrence relation

$$x_{n+2} = \frac{1}{2}(x_{n+1} + x_n) \tag{11}$$

with $a_0 = 1, a_1 = \frac{1}{2}$, and $b_0 = 0, b_1 = \frac{1}{2}$.

Proof. The case $n = 0$ is easily dealt with, since trivially we have $O_0 = 1.O_0 + 0.H_0$. To establish the other initial conditions, we appeal to (8) $O_1 = \frac{1}{2}(O_0 + H_0)$.

Now assume that for $k \leq n$, we have $O_k = a_k O_0 + b_k H_0$, with a_k and b_k as above. Then

$$\begin{aligned}
O_{n+1} &= \frac{1}{2}(O_n + H_n) \\
&= \frac{1}{2}(O_n + O_{n-1}) \\
&= \frac{1}{2}(a_n O_0 + b_n H_0 + a_{n-1} O_0 + b_{n-1} H_0) \\
&= \frac{1}{2}(a_n + a_{n-1}) O_0 + \frac{1}{2}(b_n + b_{n-1}) H_0 \\
&= \qquad a_{n+1} O_0 \quad + \quad b_{n+1} H_0
\end{aligned}$$

as required. □

Thus the sequence of circumcentres $O_1, O_2, O_3, \ldots$ corresponds to the points $O_0, \frac{1}{2}O_1 + \frac{1}{2}H_1, \frac{3}{4}O_0 + \frac{1}{4}H_0, \frac{5}{8}O_0 + \frac{3}{8}H_0, \frac{11}{16}O_0 + \frac{5}{16}H_0, \ldots$ We recognize that the numerators a_n and b_n of the coefficients of O_0 and H_0 in this sequence are related to the Jacobsthal numbers $0, 1, 1, 3, 5, 11, 21, \ldots$ We shall make this relationship clear shortly.

Proposition 3.

$$a_n = \frac{2}{3} + \frac{1}{3}(-\frac{1}{2})^n, \quad b_n = \frac{1}{3} - \frac{1}{3}(-\frac{1}{2})^n. \tag{12}$$

Proof. The characteristic equation of the recurrence (11) is

$$x^2 - \frac{1}{2}x - \frac{1}{2} = 0. \tag{13}$$

The general solution of this is $c_n = A.1^n + B.(-\frac{1}{2})^n$. Fitting the initial conditions (11) yields the result. □

Corollary 4. *The circumcentres of the sequence of triangles T_n converges to the point $\frac{2}{3}O_0 + \frac{1}{3}H_0$ on the Euler line of T.*

Proof.

$$O_n = [\frac{2}{3} + \frac{1}{3}(-\frac{1}{2})^n]O_0 + [\frac{1}{3} - \frac{1}{3}(-\frac{1}{2})^n]H_0. \tag{14}$$

Taking limits as $n \to \infty$ yields the result. □

The manner of convergence is of interest in itself, as successive points oscillate about the limit point, with each point remaining on the Euler line.

We observe that $a_n + b_n$=1, which implies that each centre O_n is a convex combination of O_0 and H_0, and hence lies on the Euler line. Further easily established relations between the two sequences are $a_n - b_n = \frac{1}{3} + \frac{2}{3}(-\frac{1}{2})^n$ and $2b_{n+1} = a_n$. This last equality comes about since

$$b_{n+1} = \frac{1}{3} - \frac{1}{3}(-\frac{1}{2})^{n+1} \Rightarrow 2b_{n+1} = \frac{2}{3} - \frac{2}{3}(-\frac{1}{2})^{n+1} = \frac{2}{3} + \frac{1}{3}(-\frac{1}{2})^n \tag{15}$$

As objects of independent study, these sequences can lead to some interesting identities. Some of these are detailed below.

Proposition 5.

$$\sum_{k=0} C(n,2k)3^{2k} = \frac{1}{2}(4^n + (-1)^n 2^n). \tag{16}$$

Proof.

$$a_n = \frac{2}{3}\left(\frac{1}{4}(1+3)\right)^n + \frac{1}{3}\left(\frac{1}{4}(1-3)\right)^n$$

$$\begin{aligned} &= \frac{1}{3}\left(\frac{1}{4}(1+3)\right)^n + \frac{1}{3}\left(\frac{1}{4}(1+3)\right)^n + \frac{1}{3}\left(\frac{1}{4}(1-3)\right)^n \\ &= \frac{1}{3} + \frac{2}{3}\frac{1}{4^n}\left(1 + \binom{n}{2}3^2 + \binom{n}{4}3^4 + ...\right) \\ &= \frac{1}{3} + \frac{1}{3}\left(-\frac{1}{2}\right)^n \end{aligned}$$

Solving between the last two lines for

$$\left(1 + \binom{n}{2}3^2 + \binom{n}{4}3^4 + ...\right)$$

yields the result. □

Note that this is [Sloane, A003665]. It is the binomial transform of the expansion of $\cosh(3x)$.

We also note that this gives us the expression

$$a_n - \frac{1}{3} = \frac{2}{3}\frac{1}{4^n}\sum_{k=0} C(n,2k)3^{2k}. \tag{17}$$

In a similar manner, we obtain

Proposition 6.

$$b_n = \frac{2}{3.4^n}\sum_{k=0} C(n,2k+1)3^{2k+1}. \tag{18}$$

Proof. We have $b_n = \frac{1}{3}(\frac{1}{4}(1+3))^n - \frac{1}{3}(\frac{1}{4}(1-3))^n$. Expanding both binomials and cancelling terms gives us the result. □

This may also be written as $b_n = \frac{2}{4^n}\sum_{k=0} C(n,2k+1)3^{2k}$. Noting that $\sum_{k=0} C(n,2k+1) = 2^{n-1}$, we can rearrange this to express b_n as $\frac{1}{2^n}$ times a weighted average of powers of 9.

$$\begin{aligned} b_n = \frac{1}{2^n}&\Big(\frac{1}{2^{n-1}}\sum_{k=0} C(n,2k+1)3^{2k}\Big) \\ &= \frac{1}{2^n}\Big(\sum_{k=0} C(n,2k+1)9^k\Big) \Big/ \sum_{k=0} C(n,2k+1\Big) \end{aligned} \tag{19}$$

We now return to exploring the link with the Jacobsthal numbers $0, 1, 1, 3, 5, 11, 21, \ldots$, with defining recurrence

$$J_{n+2} = J_{n+1} + 2J_n, J_0 = 0, J_1 = 1. \tag{20}$$

Proposition 7.

$$a_n = \frac{J(n+1)}{2^n}, \quad b_n = \frac{J(n)}{2^n}. \tag{21}$$

Proof. The recurrence (20) yields the expression $J(n) = \frac{1}{3}2^n + \frac{1}{3}(-1)^n$. Dividing through by 2^n, we obtain $\frac{J(n)}{2^n} = \frac{1}{3} + \frac{1}{3}\frac{(-1)^n}{2^n} = \frac{1}{3} + \frac{1}{3}(-\frac{1}{2})^n = b_n$. Since $a_n = 2b_{n+1}$, we get $a_n = 2\frac{J(n+1)}{2^{n+1}} = \frac{J(n+1)}{2^n}$. □

Corollary 8.

$$O_n = \frac{J(n+1)}{2^n}O_0 + \frac{J(n)}{2^n}H_0 = \frac{1}{2^n}(J(n+1)O_0 + J(n)H_0). \tag{22}$$

Corollary 9.

$$J(n) = \frac{1}{2^{n-1}} \sum_{k=0} C(n, 2k+1)3^{2k}. \tag{23}$$

Proof.

$$\begin{aligned} J(n) = 2^n b_n = \frac{2^n}{3}\frac{2}{4^n}\sum_{k=0} C(n, 2k+1)3^{2k+1} \\ = \frac{1}{2^{n-1}}\sum_{k=0} C(n, 2k+1)3^{2k} \end{aligned}$$

□

Noting again that $2^{n-1} = \sum_{k=0} C(n, 2k+1)$, we see that the last result exhibits $J(n)$ as a weighted average of even powers of 3. For instance, $J(5) = \frac{5+10.3^2+1.3^4}{5+10+1} = 11$.

Corollary 10.

$$J(n) = \frac{1}{3}(2^{n-1} + \frac{1}{2^n}\sum_{k=0} C(n-1, 2k)3^{2k}). \tag{24}$$

Proof. (17) and (21) together show that

$$\begin{aligned} J(n+1) = \frac{2^n}{3} + \frac{2}{3}\frac{1}{2^n}\sum_{k=0} C(n, 2k)3^{2k} \\ = \frac{2}{3}.2^{n-1} + \frac{1}{3}\sum_{k=0} C(n-1, 2k)3^{2k}. \end{aligned}$$

Changing from $n+1$ to n, we get $J(n) = \frac{2^{n-1}}{3} + \frac{1}{3}\frac{1}{2^{n-2}}\sum_{k=0} C(n-1, 2k)3^{2k}$, from which the result follows. □

We can rewrite this as a convex combination

$$J(n) = \frac{2}{3}2^{n-2} + \frac{1}{3}\frac{1}{2^{n-2}}\sum_{k=0} C(n-1,2k)3^{2k}.$$

We note that the last term of this expression is again a weighted average of even powers of 3, since $2^{n-2} = \sum_{k=0} C(n-1,2k)$. The numbers represented by the expression $\frac{1}{2^{n-2}}\sum_{k=0} C(n-1,2k)3^{2k}$ are the Jacobsthal–Lucas numbers $2, 1, 5, 7, 17, 31, 65, 127, \ldots$ [Sloane, A014551]. Starting at 1, they are all of the form $2^n \pm 1$.

We now consider links between the Jacobsthal numbers and Pascal's triangle. Pascal's triangle is usually represented in triangular array fashion as

```
                1
               1 1
              1 2 1
             1 3 3 1
            1 4 6 4 1
          1 5 10 10 5 1
         1 6 15 20 15 6 1
       1 7 21 35 35 21 7 1
     1 8 28 56 70 56 28 8 1
```

While there are numerous links between the Jacobsthal numbers and this number triangle, the following observations will motivate the current investigation. First, we consider

```
                -
               - 1
              1 - -
             - - 3 -
            - 4 - - 1
          1 - - 10 - -
         - - 15 - - 6 -
       - 7 - - 35 - - 1
     1 - - 56 - - 28 - -
```

Here, the '-' entry can be taken to stand for 0. Row sums of this new triangle are $0, 1, 1, 3, 5, 11, 21, 43, 85, \ldots$ In other words, we have the beginning of the sequence of Jacobsthal numbers. Similarly, the following modified triangle

```
            -
           1 -
          - - 1
         - 3 - -
        1 - - 4 -
       - - 10 - - 1
      - 6 - - 15 - -
     1 - - 35 - - 7 -
    - - 28 - - 56 - - 1
```

yields the same sequence of numbers $0, 1, 1, 3, 5, 11, 21, \ldots$ For completeness, a look at what is 'left over' is also informative.

```
            1
           - -
          - 2 -
         1 - - 1
        - - 6 - -
       - 5 - - 5 -
      1 - - 20 - - 1
     - - 21 - - 21 - -
    - 8 - - 70 - - 8 -
```

This gives us the sequence $1, 0, 2, 2, 6, 10, 22, 42, 86, \ldots$ We note that these numbers are of the form $J_n \pm 1$. They form the start of the sequence [Sloane, A078008]. For the purposes of this article, we shall define $J'_n = J_n + (-1)^n$. We then have $J'(n) = \frac{2^n}{3} + \frac{2(-1)^n}{3}$. $J'(n)$ is a solution to recurrence (1), with initial conditions $a_0 = 1, a_1 = 0$. $J'(n) = J(n+1) - J(n) = 2^n(a_n - b_n)$.

Recall now that the sum of the rows of Pascal's triangle are of the form 2^n, a consequence of the well-known identity

$$\sum_{k=0}^{n} C(n,k) = 2^n. \tag{25}$$

The above results suggest the following decomposition of 2^n (and figuratively, a decomposition of Pascal's triangle).

$$2^0 = 1 = 0 + 0 + 1 \tag{26}$$
$$2^1 = 2 = 1 + 1 + 0 \tag{27}$$
$$2^2 = 4 = 1 + 1 + 2 \tag{28}$$
$$2^3 = 8 = 3 + 3 + (1 + 1) \tag{29}$$
$$2^4 = 16 = 5 + 5 + 6 = (1 + 4) + (4 + 1) + 6 \tag{30}$$
$$2^5 = 32 = 11 + 11 + 10 = (1 + 10) + (10 + 1) + (5 + 5) \tag{31}$$

This leads to:

Proposition 11.

$$2^n = 2J_n + J'_n. \tag{32}$$

Proof. $2J_n + J'_n = 2(\frac{2^n}{3} - \frac{(-1)^n}{3}) + (\frac{2^n}{3} - \frac{(-1)^n}{3}) + (-1)^n = 2^n$. □

Corollary 12.

$$2^n = J(n) + J(n+1). \tag{33}$$

We now wish to show that this provides a decomposition for the rows of Pascal's triangle, as indicated by the equations (26)–(31) above.

Proposition 13.

$$J(n) = \sum_{(n+k) \mod 3=1} C(n,k) = \sum_{(n+k) \mod 3=2} C(n,k), \tag{34}$$

$$J'(n) = \sum_{(n+k) \mod 3=0} C(n,k). \tag{35}$$

Proof. We let

$$a(n) = \sum_{(n+k) \mod 3=1} C(n,k) \quad \text{and} \quad a'(n) = \sum_{(n+k) \mod 3=2} C(n,k).$$

We wish to show that $a(n) = a'(n) = J(n)$. We start by establishing the initial conditions. $a(0) = \sum_{(n+k) \mod 3=1} C(0,k) = 0$ since only $C(0,0) \neq 0$. $a(1) = \sum_{(n+k) \mod 3=1} C(1,k) = C(1,1) = 1$ since all other terms $C(1,k)$ with $(n+k) \mod 3 = 1$ have value 0. We can similarly show that $a'(0) = 0, a'(1) = 1$. □

We now wish to establish that $a(n+2) = a(n+1) + 2a(n)$. For this, we employ the following lemma.

Lemma 14.

$$\begin{aligned} a(n+2) &= a'(n+1) + a'(n) + a(n) && (36) \\ a'(n+2) &= a(n+1) + a(n) + a'(n) && (37) \end{aligned}$$

Proof. We have

$$\begin{aligned} a(n+2) &= \sum_{(n+2+k) \mod 3=1} C(n+2,k) = \sum_m C(n+2,3m-n-1) \\ &= \sum_m (C(n+1,3m-n-2) + C(n+1,3m-n-1)) \\ &= \sum_m C(n+1,3m-n-2) \\ &\qquad + \sum \big(C(n,3m-n-1) + C(n,3m-n-2)\big) \\ &= \sum_{(n+1+k) \mod 3=2} C(n+1,k) + \sum_{(n+k) \mod 3=2} C(n,k) \\ &\qquad + \sum_{(n+k) \mod 3=1} C(n,k) \\ &= a'(n+1) + a'(n) + a(n) \end{aligned}$$

In a similar fashion, we have

$$\begin{aligned} a'(n+2) &= \sum_{(n+2+k) \mod 3=2} C(n+2,k) = \sum_m C(n+2,3m-n) \\ &= \sum_m (C(n+1,3m-n-1) + C(n+1,3m-n)) \\ &= \sum_m C(n+1,3m-n-1) \\ &\qquad + \sum \big(C(n,3m-n-2) + C(n,3m-n-1)\big) \\ &= \sum_{(n+1+k) \mod 3=1} C(n+1,k) + \sum_{(n+k) \mod 3=1} C(n,k) \\ &\qquad + \sum_{(n+k) \mod 3=2} C(n,k) \\ &= a(n+1) + a(n) + a'(n) \end{aligned}$$

The proof of the first assertion now follows from the observation that $a(n+2) - a'(n+2) = a'(n+1) + a'(n) + a(n) - a(n+1) - a(n) - a'(n)$

and

$$a(n+2) - a'(n+2) = a'(n+1) - a(n+1). \tag{38}$$

Hence,

$$\begin{aligned} a(n+2) - a'(n+2) = a'(n+1) - a(n+1) = a(n) - a'(n) \\ = \ldots = 1 - 1 = 0. \end{aligned}$$

Thus $a(n) = a'(n)$ and so $a(n+2) = a'(n+1) + a'(n) + a(n) = a(n+1) + 2a(n)$ as required. In order to prove the second assertion, we make use of the fact that $J'(n) = J(n+1) - J(n)$. We then have

$$\begin{aligned} J'(n) &= J(n+1) - J(n) \\ &= \sum_{(n+1+k) \mod 3=1} C(n+1,k) - \sum_{(n+k) \mod 3=2} C(n,k) \\ &= \sum_m C(n+1, 3m-n) - \sum_{(n+k) \mod 3=2} C(n,k) \\ &= \sum_m C(n, 3m-n) + \sum_m C(n, 3m-n-1) \\ &\qquad - \sum_{(n+k) \mod 3=2} C(n,k) \\ &= \sum_m C(n,3m-n) + \sum_{(n+k) \mod 3=2} C(n,k) - \sum_{(n+k) \mod 3=2} C(n,k) \\ &= \sum_{(n+k) \mod 3=0} C(n,k). \end{aligned}$$

□

There is in fact a second Jacobsthal decomposition of Pascal's triangle, based on the fact that $2^n = J(n) + J(n+1)$. The following display makes this evident.

$$\begin{array}{c} 1 \\ \underline{1}\ 1 \\ 1\ 2\ \underline{1} \\ 1\ 3\ \underline{3}\ 1 \\ \underline{1}\ 4\ 6\ \underline{4}\ 1 \\ 1\ 5\ \underline{10}\ 10\ 5\ \underline{1} \\ 1\ \underline{6}\ 15\ 20\ \underline{15}\ 6\ 1 \\ \underline{1}\ 7\ 21\ \underline{35}\ 35\ 21\ 7\ 1 \\ 1\ 8\ \underline{28}\ 56\ 70\ \underline{56}\ 28\ 8\ \underline{1} \end{array}$$

Here, underlined elements sum to $J(n)$ and non-underlined elements sum to $J(n+1)$.

We end with an observation that follows from an examination of the 'Jacobsthal' triangles. We recall that the Fibonacci numbers [Sloane, A000045] can be obtained as the sums of the diagonals of Pascal's triangle. The above proposition provides us with a decomposition of Pascal's triangle that effectively tri-sects the Fibonacci numbers: the diagonals of the triangles shown give us F(3n), F(3n+1) and F(3n+2) [Sloane, A001076, A033887, A015448].

Concerned with sequences A000045, A001045, A001076, A007318, A014551, A015448, A033887, A078008.

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