Rank Rigidity and Symmetry

J.-H. ESCHENBURG

ABSTRACT. Symmetric spaces form a subject of central importance in geometry. We report on a class of theorems characterizing Riemannian symmetric spaces of higher rank. These theorems shed light on various aspects of these spaces and bring together very different geometric theories: Riemannian manifolds of nonpositive curvature, spherical Tits buildings and submanifold geometry. In particular, we put emphasis on the third aspect which is related to isotropy orbits of symmetric spaces.

1. The Desarguesian property

There are several theorems in geometry where a lower bound on a certain dimension ("rank") together with other geometric properties determines a space uniquely up to automorphisms ("rigidity"). The automorphism group of such a space is very large; in fact rigidity is proved by detecting sufficiently many automorphisms. The rank assumption is remarkable; it contradicts our feeling that higher dimension should imply more flexibility and less rigidity. However the geometric assumptions yield more and more substructures as the rank grows, and these are used for the rigidity proof. A classical example is the theorem of Desargues in n-dimensional projective geometry $\mathbf{P}^n$:

(K): Desargues, Kolmogoroff [K]:
If $\mathbf{P}^n$ is connected and compact with $n \geq 3$, then $\mathbf{P}^n$ is projective n-space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$
(false for $n = 2$).

The theorem can be proved by showing that the automorphisms fixing an arbitrary hyperplane act transitively on its complement. The hyperplane can be viewed as hyperplane "at infinity", so its complement is affine space $\mathbf{A}^n$. For any two points $p, p' \in \mathbf{A}^n$ we may define the translation sending p to p' provided that the "small" Desargues theorem is satisfied: If the lines k, l, m are parallel then $a \,||\, a',\; b \,||\, b' \;\Rightarrow\; c \,||\, c'$ (Fig. 1).

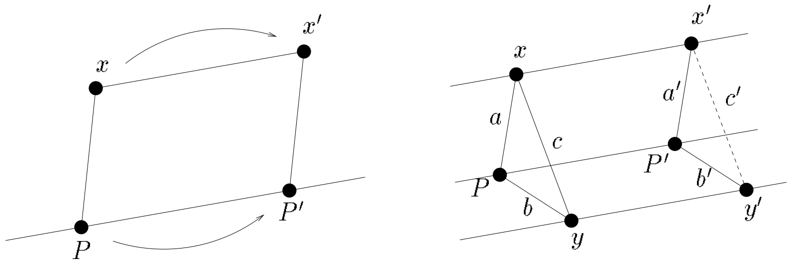

Figure 1: Translations and Desargues' property

But the Desarguesian property does not follow from the axioms of projective geometry in dimension 2; there are many projective planes lacking this property (cf.[Sal]). However, it *does* follow in dimension $n \geq 3$: If the three parallel lines k, l, m are not complanar, the two triangles abc and $a'b'c'$ in Fig. 1 are lying in two different parallel planes. Thus the lines c and c' do not intersect inside $\mathbf{A}^n$, but on the other hand they lie in a common plane (spanned by the parallel lines k and m), so they must be parallel. This is a model case for the use of substructures (planes in this case) and their intersections in order to construct automorphisms.

Projective geometry was put in a much broader context by J. Tits [Ti], leading to a far reaching generalization of (K) (cf. (A) in Ch. 3): A *spherical building* of rank n is a simplical complex of dimension $n-1$ such that any two simplices are lying in a common finite subcomplex called *appartment*. Each appartment is a triangulation of the $(n-1)$-sphere being invariant under a group of hyperplane reflections, the *Weyl group*, which acts simply transitively on the set of $(n-1)$-simplices ("chambers") of the appartment. Any two appartments are isomorphic, and two such isomorphisms differ by a Weyl group element. Simplices which are equivalent under such an isomorphism are said to have the same *type*. In the case of $\mathbf{P}^n$, the $(n-1)$-simplices are the *full flags*

$$\text{point } p \subset \text{line } l \subset \text{plane } E \subset \ldots \subset \mathbf{P}^n$$

and the faces are the corresponding subflags of all possible types (Fig. 2),

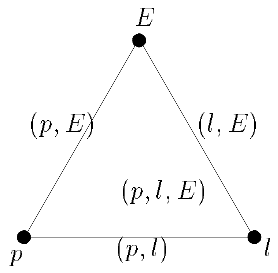

Figure 2: Flag simplex

while the appartments are the formed by all flags corresponding to the faces of a fixed projective n-simplex in $\mathbf{P}^n$ (Fig. 3).

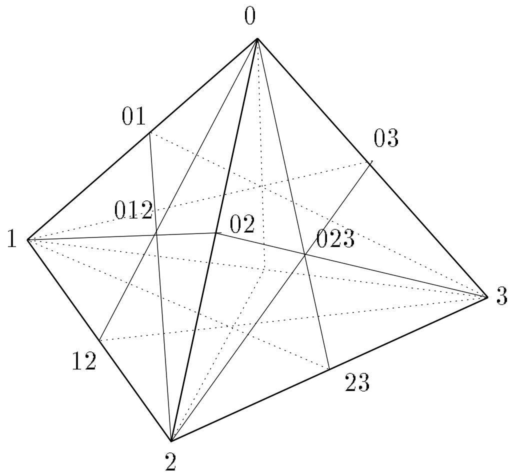

Figure 3: Appartments of $\mathbf{P^3}$

2. SYMMETRIC SPACES

A *Riemannian manifold* is, roughly speaking, a metric space which is locally approximated by euclidean geometry (on the tangent space) and where distance globally is measured by arc length of curves. Shortest curves are called *geodesics*. The deviation of Riemannian from euclidean geometry at any point x is measured by a quantity $R(x)$ called *curvature tensor* from which the notion of *sectional curvature* (sc) is derived. E.g. we have $sc \geq 0$ (resp. $sc \leq 0$) iff the distance between two geodesics emanating from a point x near x

does not grow more (resp. less) than proportional to the distance from x (Fig. 4).

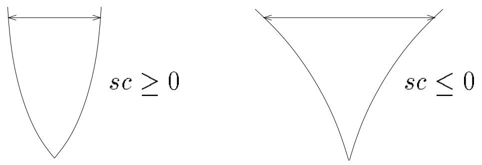

Figure 4: Distance of geodesics

Globally, the most important difference to euclidean geometry is the path dependence of the parallel displacement of vectors in Riemannian geometry. The phenomenon is measured by the *holonomy group* which consists of the parallel displacements along all loops starting and ending at a fixed point x; this is a group of linear transformations on the tangent space at x.

A *symmetric space* (cf. [Hg], [E1]) is a Riemannian manifold X which has an isometric point reflection ("symmetry") σ_x at any point $x \in X$. So there are plenty of isometries which turn X into a *homogeneous space*, i.e. the group of isometries acts transitively on X. Locally, symmetric spaces are characterized by the constancy of the curvature tensor R, more precisely, R is invariant under all parallel displacements. This partially explains the importance of these spaces for Riemannian geometry. Examples include euclidean n-space, spheres, Grassmannians (cf. Fig. 5)

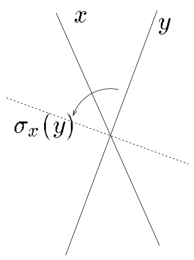

Figure 5: Symmetry of Grassmannians

and compact Lie groups (with $\sigma_x(y) = xy^{-1}x$). Symmetric spaces always come in "dual" pairs X, X^*:

$$X \text{ compact, } sc \geq 0 \;\longleftrightarrow\; X^* \text{ noncompact, } sc \leq 0$$

E.g. the dual of the sphere is hyperbolic space, the dual of the Grassmannian of n-planes in $\mathbb{R}^{n+k}$ is the set of "spacelike" n-planes with respect to the indefinite inner product of index k on $\mathbb{R}^{n+k}$, and the dual of a compact Lie group G is G^c/G where G^c is the complexification of G. The irreducible symmetric spaces of noncompact type are precisely the coset spaces $X = G/K$ where G is a simple noncompact Lie group (the identity component of the isometry group of X) and K its maximal compact subgroup. Hence irreducible symmetric spaces are labeled by the simple noncompact Lie groups.

The geometry of a symmetric space $X = G/K$ of noncompact type can be understood best by looking at the unit 2-disk (Poincaré) model of the hyperbolic plane (Fig. 6).

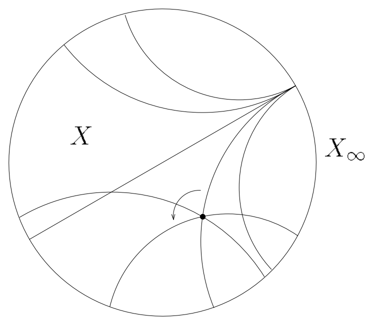

Figure 6: Hyperbolic plane

In general, X is diffeomorphic to the open n-disk, and there is a canonical way of attaching an $(n-1)$-sphere X_∞ to X, using geodesic rays emanating from any point of X; two geodesic rays starting at different points define the same point of X_∞ if they have finite distance from each other. The geodesics in the hyperbolic plane (orthocircles in Fig. 6) have to be replaced by r-dimensional totally geodesic euclidean subspaces (called *flats*) where $r \geq 1$ is called the *rank* of the symmetric space. Any two points in $\overline{X} := X \cup X_\infty$ are joined by a common flat. Two flats can join at X_∞, but the possible set of intersection is not arbitrary. In fact, there is a G-invariant decomposition of X_∞ forming a spherical building of rank r, whose appartments are the flats at X_∞, and two flats may join at X_∞ only along entire simplices of the building. E.g. the bulding of projective geometry $\mathbf{P}^{n-1}$ is associated to the symmetric space $SL(n,\mathbb{R})/SO(n)$.

A major role is played by the *isotropy group* K which replaces the rotations in euclidean space and which for irreducible symmetric spaces happens to agree with the holonomy group at the base point. In euclidean space, we may turn around and look in every direction we like, so the isotropy orbits are spheres. But in a symmetric space of rank r, the isotropy group must preserve the set of r-flats through the base point. In fact K acts transitively on this set, and a K-orbit intersects each flat a finite number of times, and each time perpendicularly (Fig. 7).

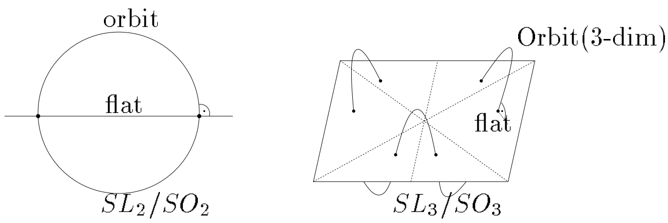

Figure 7: Isotropy orbit and flat

The isotropy action of a symmetric space, viewed as a linear action on the tangent space of the base point, is also called *s-representation*. The principal orbits of this action are examples of *isoparametric submanifolds* (cf. Ch. 4). Any K-orbit, when projected to X_∞ by the geodesic rays from the base point, is invariant also under the noncompact group G acting on X_∞, hence a so called *R-space* (a compact homogeneous space with an effective transitive action of a noncompact Lie group). It can be viewed as the set of all simplices of a fixed type in the spherical building associated to X; in fact it intersects any of these simplices precisely once. Some of these orbits are again symmetric spaces, so called *extrinsic symmetric spaces* or *symmetric R-spaces* (cf. [F], [EH1]); in the case of $\mathbf{P}^n$, these are the one-step flag manifolds, the Grassmannians. Most symmetric spaces occur in this way. It is due to this fact that spheres and projective spaces allow a transitive action by a noncompact Lie group (conformal and projective groups).

3. Some rank rigidity theorems

(A): Tits [Ti], Burns-Spatzier [BS1]:

If a locally connected irreducible compact spherical building has rank $r \geq 3$, then it is associated to a symmetric spaces of rank r

(false for $r = 2$).

(B): Berger [Be], Simons [Si]:

If the holonomy group of a locally irreducible Riemannian manifold acts with cohomogeneity (codimension of the maximal orbits) $k \geq 2$ then it is locally symmetric of rank k

(false for $k = 1$).

(C): *If a complete locally irreducible Riemannian manifolds has rank $r \geq 2$, i.e. r is the largest number such that any two points can be joined by an r-dimensional complete locally euclidean and totally geodesic submanifold ("flat"), then it is locally symmetric of rank r, provided that additionally one of the following properties holds:*

(a): Ballmann [B], Eberlein [Eb], Burns-Spatzier [BS2]:
$sc \leq 0$ and finite volume,

(b): Heber [Hb]:
$sc \leq 0$ and homogeneity,

(c): Molina-Olmos [MO]:
All flats are compact,

(d): Samiou ([Sa], also cf. [EO]):
All pointed flats are congruent under the isometry group.

((a)-(c) *are false for $r = 1$. Also for $r \geq 2$ the statement is false without additional assumptions, even for homogeneous spaces, cf.* [SS].)

(D): Thorbergsson [Th], Olmos [O2]:

A closed, irreducible, substantial isoparametric submanifold of codimension $k \geq 3$ in euclidean n-space is a principal orbit of the isotropy representation of a symmetric space of rank k

(false for $k = 2$).

There are many more such theorems which all shed light on different aspects of symmetric spaces. (A) is the direct generalization of (K) and is proved by constructing the analogue of the translations.

In (B) the holonomy group is assumed to be small, so it has many invariants, and eventually it is shown that R is holonomy invariant, in fact parallel. (A) and (B) are the main tools for the proofs of (C) and (D). In fact, (A) is used by [BS2], [Hb], [Th] while (B) and related tools occur in [B], [Eb], [MO], [Sa], [ES] and [O2]. However, the interdependence of (A) and (B) is not clear. We will now concentrate on a new proof of (D) (joint work with E. Heintze, cf. [EH2], [E2], [HL]) which uses submanifold theory. It is based on ideas of [O2] but avoids the classification of Dadok [D] used in [Th] and [O2].

4. Isoparametric submanifolds

An n-dimensional submanifold $M = M^n \subset \mathbb{R}^{n+k}$ has a tangent space $\tau_p = \tau_p M$ and a normal space $\nu_p = \nu_p M$ at any point $p \in M$ (Fig. 8).

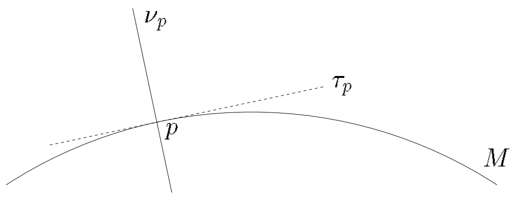

Figure 8: Tangent and normal spaces

A normal vector field ξ on M is called *parallel* if $\partial_v \xi \in \tau_p$ for all $v \in \tau_p$. The self adjoint linear map $(\partial\xi)_p$ on τ_p (the *shape operator*) measures how the normal space ν_p changes when p is moved; its eigenvalues are the *principal curvatures*. A (closed irreducible substantial) submanifold $M^n \subset \mathbb{R}^{n+k}$ is called *isoparametric* (cf. [Se], [PT]) iff any normal vector can be extended to a parallel normal field ξ, and the eigenvalues of $(\partial\xi)_p$ are independent of p. There are two classes of examples known:

- **I:** Principal orbits of s-representations (*s-orbits*),
- **II:** hypersurfaces in spheres related to Clifford representations (cf. [FKM]).

The classification depends on the codimension k:

- $k = 1$**:** only spheres (cf. [Se])
- $k = 2$**:** open conjecture: only I and II ?
- $k \geq 3$**:** only I by (D).

Isoparametric submanifolds behave in many ways like principal group orbits. They generate families of submanifolds $M_\xi = \{p + \xi_p;\ p \in M\}$ where ξ is any parallel normal field, and these foliate the whole space $\mathbb{R}^{n+k}$. In fact M_ξ is again isoparametric and diffeomorphic to M for most ξ, but not for all; some *singular* M_ξ have lower dimension, like singular group orbits. However what is special to isoparametric submanifolds and not true for arbitrary group orbits (in fact by [D] this holds only for s-orbits): the normal spaces are common to all leaves M_ξ, and two normal spaces may intersect only in points of singular leaves. Fig. 9 gives a picture in the (however non-substantial) situation where M is a planar circle in 3-space:

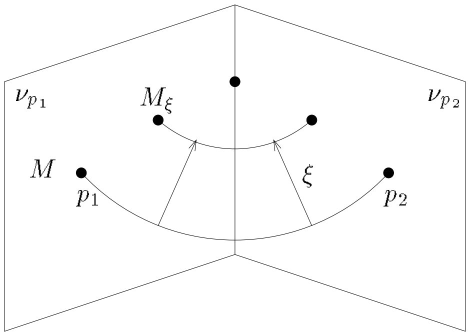

Figure 9: Common normal spaces

How does $k \geq 3$ imply homogeneity? The main idea is again to pass to substructures, namely the fibres of the projections onto singular leaves: $\pi = \pi_\xi : M \to M_\xi$, $\pi_\xi(p) = p + \xi_p$ where $M_\xi = S$ is singular. Let $q = \pi(p) \in S$ and $F = \pi^{-1}(q) = M \cap \nu_q S$ the fibre through p (Fig. 10).

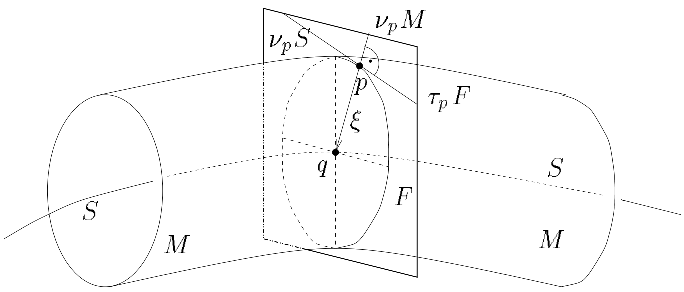

Figure 10: Projection onto a singular leaf

(In the $\mathbf{P}^3$-case, M corresponds to the full flag manifold and S to a partial flag manifold, say the set of all planes, and π maps a flag (p, l, E) onto E. Then F corresponds to the flags inside the plane E.)

We show first that F is *totally geodesic* in M, i.e. any geodesic in F is a geodesic also in M. A curve in a submanifold is a geodesic iff its second derivative is always in the normal space. Now F is the intersection of M with the linear space $\nu_q S$, thus a geodesic $p(t)$ in F satisfies $p'' \in \nu_p F \cap \nu_q S$. But since M is isoparametric, $\nu_p M$ lies in $\nu_q S$ with orthogonal complement $\tau_p F$ (cf. Fig. 10). Hence $\nu_p F \cap \nu_q S = \nu_p M$, and $p(t)$ is a geodesic in M as well.

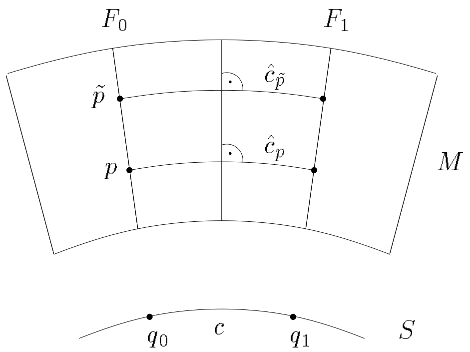

Figure 11: Horizontal lifts

Thus $\pi : M \to S$ is a projection (submersion) with totally geodesic fibres. This implies homogeneity of the fibres as follows. Take any two points $q_0, q_1 \in S$ and connect them by a smooth curve $c : [0,1] \to S$. Let $F_i = \pi^{-1}(q_i)$. For any $p \in F_0$ there is a unique curve $\hat{c}_p$ on M starting from p and being a lift of c, i.e. $\pi \circ \hat{c}_p = c$, such that $\hat{c}_p'(t)$ is perpendicular to the fibre over $c(t)$ at all t (*horizontal lift*, cf. Fig. 11). Thus we have defined a map $h_c : F_0 \to F_1$, $p = \hat{c}_p(0) \mapsto \hat{c}_p(1)$. Now π has geodesic fibres iff all h_c are isometries (cf. Fig. 11). In particular, all closed curves in S starting and ending at $q \in S$ give rise to isometries of the fibre F over q, turning F into a homogeneous space. In fact, these isometries form the *normal holonomy group* of S describing parallel displacements of normal vectors on S, and by an observation of Olmos [O1] (which is related to (B) but much simpler), the action of the normal holonomy group is always an s-representation. If $k \geq 3$, there is a whole hierarchy of such fibrations π_ξ, becoming more and more singular. In fact, for any two "simply" singular normal vectors $\xi_1, \xi_2 \in \nu_p M$ we find a "doubly" singular $\xi \in \nu_p M$ such that the fibres F_1, F_2, F of $\pi_{\xi_1}, \pi_{\xi_2}, \pi_\xi$ through p satisfy $F_1, F_2 \subset F$. This makes the s-orbit structures on F_1 and F_2 compatible and eventually leads to an s-orbit structure on M. (Technically, the curvature tensor of the corresponding symmetric space is constructed on $\mathbb{R}^{n+k}$ using the curvature tensors on the various normal spaces $\nu_q S$; these are given by Olmos [O1].)

5. Infinitesimal rank rigidity

There is yet another classical rank rigidity theorem which seems to be of very different type:

(L): (Liouville) ([L], cf. [Sp])
If U, V are open subsets of the n-sphere S^n with $n \geq 3$ and $\phi : U \to V$ is a smooth conformal diffeomorphism, then ϕ is the restriction to U of an element of the conformal diffeomorphism group of S^n (which is $O^+(n+1,1)$)
(false for $n = 2$).

Unlike in (A)-(D), the geometric assumption here is infinitesimal: The derivative of ϕ must be a conformal linear map (orthogonal up to rescaling). Recently, Liouville's theorem was generalized and put in the framework of symmetric R-spaces by W. Bertram [Bt]. We do not know if (L) can be also related to (A)-(D). But we could prove a rigidity theorem of a similar type whose proof is closely related to (B) and (D) (cf. [E3]). The map ϕ is replaced by a submanifold M of a symmetric space, and the assumption on the derivative of ϕ is turned into a condition for the tangent and normal spaces of M:

(E): *Let X be an irreducible symmetric space of compact type and $M \subset X$ a submanifold having totally geodesic tangent and normal subspaces of X at any of its points. Suppose further that all tangent subspaces are irreducible and of rank $r \geq 2$. Then M is an extrinsic symmetric isotropy orbit in X.*
(false for $r = 1$).

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Jost-Hinrich Eschenburg,
Institut für Mathematik,
Universität Augsburg,
D–86135 Augsburg, Germany
eschenburg@math.uni-augsburg.de

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