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Elementary Operators on Calkin Algebras

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This paper is dedicated to the memory of Professor Klaus Vala.

Properties of elementary operators, especially on algebras of operators, have been vigorously studied over the past decades. It emerges that the situation of the Calkin algebra is full of surprises. In this setting, an elementary operator with dense range is surjective, and injectivity entails boundedness below. In the case of Hilbert space, positivity implies complete positivity, and the norm and the *cb*-norm of every elementary operator coincide. We present an overview on some results of this flavour, in particular on recent extensions of the latter two results to antiliminal-by-abelian C^*-algebras (obtained by Archbold, Somerset, and the author), and strong rigidity properties of the norm of elementary operators on Calkin algebras due to Saksman and Tylli.

1. Introduction

Properties of elementary operators have been investigated during the past two decades under a variety of aspects. There are detailed studies of their spectra, see [**8**] for an overview, a number of works were devoted to structural questions, such as compactness and norm properties, cf. [**9**], and also their compatibility with various order relations has been closely examined. Through all these studies it emerged that, for general elementary operators, a full description of their properties is rather intricate since these are often intimately interwoven with the structure of the underlying algebras.

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It thus comes as a surprise that, on algebras which have an apparently complicated structure themselves, like the Calkin algebra on a Banach space and related algebras, almost all of the complications that one encounters in the general setting seem to disappear and the theory runs extremely smoothly. It is the goal of this survey to highlight a number of instances where this can be observed and to try to underline the properties of the Calkin algebra which can be exploited in each case to obtain the neatest results.

In following Grothendieck in his programme to study operators on Banach spaces via tensor products, we will introduce elementary operators by a canonical mapping from the algebraic tensor product of the underlying Banach algebra with itself. In fact, elementary operators do feature prominently in various approximation results for more complicated classes of mappings!

In the sequel, A will denote a (complex) unital Banach algebra. Let $L(E)$ stand for the Banach algebra of all bounded linear operators on a Banach space E. On the algebraic tensor product $A \otimes A$, the following canonical mapping Θ is defined through

$$\Theta\colon A \otimes A \to L(A), \quad a \otimes b \mapsto M_{a,b},$$

where $M_{a,b}x = axb$ for some $a, b \in A$. The image of Θ is denoted by $\mathcal{E}\ell(A)$ and called *the algebra of elementary operators on A*. Thus, an element of $\mathcal{E}\ell(A)$ is a linear mapping of the form

$$S\colon x \mapsto \sum_{j=1}^{n} a_j x b_j \qquad (x \in A).$$

Remark 1.1. There are various other settings for the definition of elementary operators. For instance, if A is non-unital, then it is more natural to take the coefficients a_j, b_j from the multiplier algebra of A rather than A. An axiomatic approach to elementary operators has been proposed in [**6**, **7**].

In infinite dimensions, Θ cannot be expected to be surjective, but often, the image of Θ is rich enough to describe other interesting classes of operators. For example, the norm closure of $\mathcal{E}\ell(K(H))$ coincides with $K(K(H))$; here, $K(E)$ stands for the algebra of compact operators on a Banach space E and H denotes a Hilbert space (throughout). On the other hand, injectivity of Θ is closely related to the algebraic structure of A. Let us suppose, for a moment, that Θ *is* one-to-one so that $A \otimes A \cong \mathcal{E}\ell(A)$. Then, the operator norm induces a norm on $A \otimes A$ but to get a cross norm, we must have $\|M_{a,b}\| = \|a \otimes b\| = \|a\|\,\|b\|$ for all $a, b \in A$. This has severe consequences for A, namely A has to be *ultraprime* and, in particular, the centre $Z(A)$ of A has to be trivial. For more details on this we refer the reader to [**17**] and confine ourselves here with the remark that, in order to relate elementary operators fully to tensor products, we need to consider module tensor products (over $Z(A)$ or related commutative algebras) rather than vector space tensor products. For an initial discussion how to use primeness of C^*-algebras in describing properties of elementary operators, see [**19**].

Let E be a Banach space. By $C(E) = L(E)/K(E)$ we shall, as is customary, denote the *Calkin algebra* on E (although $C(E)$ does not act on E). We will now start our discussion of properties of elementary operators on Calkin algebras.

2. Spectral Properties

Let us begin with a very basic observation. Suppose that the two-sided multiplication $M_{a,b}$ has dense range. Since the group of invertible elements in A is open, there thus exists $x \in A$ such that axb is invertible. Consequently, b has a left inverse, say b'. Similarly, a has a right inverse, say a'. It follows that $M_{a,b}M_{a',b'} = \mathrm{id}_A$ wherefore $M_{a,b}$ is surjective.

This nice property, however, does not extend to more general elementary operators. For instance, it fails for a generalised inner derivation $\delta_{a,b}\colon x \mapsto ax - xb$ even on such 'good' algebras as $A = L(H)$. The picture changes drastically once we consider the Calkin algebra.

Theorem 2.1. [**12**] *Let $A = C(\ell^2)$, and let $S = \sum_{j=1}^{n} M_{a_j,b_j}$ for some commutative subsets $\{a_j\}$, $\{b_j\}$ in A. Then S is surjective if it has dense range and S is bounded below if it is injective.*

What is the property of the Calkin algebra that is used in this result? On the one hand, a close relation between the left essential spectrum, the approximate point spectrum and the boundary of the spectrum which was observed by Fillmore, Stampfli and Williams

[**10**]. On the other hand, an effective use of the primeness of a C^*-algebra allows a full description of the spectrum of an arbitrary elementary operator, see [**18**, Part I]. Putting these two together we obtain an extension of Gravner's theorem, in which the notation is as follows. For $T \in L(A)$ we let

$$\begin{aligned}
\sigma_d(T) &= \{\lambda \in \mathbb{C} \mid \lambda - T \text{ is not surjective}\},\\
\sigma_c(T) &= \{\lambda \in \mathbb{C} \mid \lambda - T \text{ does not have dense range}\},\\
\sigma_\pi(T) &= \{\lambda \in \mathbb{C} \mid \lambda - T \text{ is no topological isomorphism}\},
\end{aligned}$$

and

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid \lambda - T \text{ is not injective}\}$$

denote the *defect*, *compression*, *approximate point*, and *point spectrum*, respectively.

Theorem 2.2. [**17**] *Let A be a unital prime C^*-algebra such that every topological divisor of zero is a divisor of zero. Suppose that $S = \sum_{j=1}^n M_{a_j,b_j}$ for some commutative subsets $\{a_j\}$, $\{b_j\}$ in A. Then*

$$\sigma_d(S) = \sigma_c(S) \qquad \textit{and} \qquad \sigma_\pi(S) = \sigma_p(S).$$

More recently, Saksman and Tylli took up the situation of the Calkin algebra on Banach spaces which are no Hilbert spaces and obtained the following generalisation of Theorem 2.1. Note, in particular, that all commutativity assumptions on the sets of coefficients have now disappeared.

Theorem 2.3. [**23**] *Let $A = C(\ell^p)$, where $1 < p < \infty$, and let $S \in \mathcal{E}\ell(A)$. Then S is non-surjective only if $A/\overline{\operatorname{im} S}$ is non-separable and S is not bounded below only if $\ker S$ is non-separable.*

Of course, their result implies that $\sigma_d(S) = \sigma_c(S)$ and $\sigma_\pi(S) = \sigma_p(S)$, but in a much stronger form.

3. Compactness Properties

An element a in a Banach algebra A is called *compact* if the multiplication $M_{a,a}$ is a compact operator on A. This concept was introduced by Vala [**28**] in 1967 since, if $A = L(\ell^2)$ then this amounts to the requirement that $a \in K(\ell^2)$. In 1975 Ylinen showed that this is equivalent to the weak compactness of $M_{a,a}$. In [**11**], Fong and Sourour characterised compact elementary operators on $L(H)$; from their description, the following conjecture arose:

There is no non-zero compact elementary operator on the Calkin algebra on a (separable) Hilbert space.

In the sequel, this conjecture was tackled from various viewpoints. In [**1**], Apostol and Fialkow confirmed the conjecture by establishing a strong rigidity property of the essential norm, see also the next section. Magajna [**14**] exploited the simplicity of $C(\ell^2)$ together with some algebraic arguments to come up with an alternative proof. Using the primeness of the Calkin algebra, the fact that $C(H)$ does not contain any non-zero compact element, and some representation theory, the following extension of the Fong-Sourour conjecture was then found.

Theorem 3.1. [**18**, Part II] *Let A be a C^*-algebra without any non-zero compact elements. Then every weakly compact elementary operator on A must vanish.*

This somewhat stronger result initiated the generalised Fong-Sourour conjecture which asks for a characterisation of those Banach spaces E for which there are no non-zero weakly compact elementary operators on $C(E)$. Very recently, Saksman and Tylli made an important contribution towards this question by taking up the ideas of Apostol and Fialkow from [**1**] and developing them much further. They obtained the following result.

Theorem 3.2. [**24**] *Let $A = C(\ell^p)$, where $1 < p < \infty$. Then every weakly compact elementary operator on A vanishes.*

This theorem in fact holds in a somewhat wider setting, which we shall now proceed to discuss.

4. Norm Properties

In [**1**] it was shown that the norm of every elementary operator S on $C(\ell^2)$ agrees with the distance from S to the compact operators on $C(\ell^2)$. In this way, the Fong-Sourour conjecture was confirmed. The proof by Apostol and Fialkow rests on the non-commutative Weyl-von Neumann theorem.

Using a more Banach space geometric approach and a delicate gliding-hump argument, Saksman and Tylli extended this result to Calkin algebras on Banach spaces with 1-unconditional basis; indeed, they proved the following theorem.

Theorem 4.1. [**24**] *Let E be a Banach space with a 1-unconditional basis. For every $S \in \mathcal{E}\ell(C(E))$, we have*

$$\|S\| = \|S\|_e = \|S\|_w,$$

where $\|\cdot\|_e$ and $\|\cdot\|_w$ denote the essential and the weak essential norm, respectively.

Clearly, this result immediately yields Theorem 3.2.

There is no general formula known describing the norm of an arbitrary elementary operator, even for such 'simple' algebras as $K(H)$. (For a survey on the state-of-the-art of this problem, see [**22**].) Already the case of an inner derivation on a C^*-algebra took some time to be settled; see [**3**; Section 4.6] for an overview of the history of this problem. Once again, the situation of the Calkin algebra appears to be an exception. Combining results in [**15**] and [**2**], we obtain the following quite satisfactory description.

Theorem 4.2. *Let S be an elementary operator on $C(H)$. Then*

$$\|S\| = \inf\Big\{\Big\|\sum_{j=1}^{n} a_j a_j^*\Big\|^{1/2}\,\Big\|\sum_{j=1}^{n} b_j^* b_j\Big\|^{1/2}\Big\},$$

where the infimum is taken over all representations of S as $S = \sum_{j=1}^{n} M_{a_j,b_j}$.

The key to this result is, on the one hand, the Haagerup tensor norm $\|\cdot\|_h$ on $A \otimes A$, which has exceptional properties for every C^*-algebra A. These, in particular, enabled us to prove that Θ is an isometry from $A \otimes_h A$ into the space of all completely bounded operators on A, provided that A is a prime C^*-algebra [**2**]. And, on the other hand, the fact that $\|S\|$ and $\|S\|_{cb}$, the cb-norm of S, coincide for $A = C(H)$ [**15**]. This clearly raises the question for which class of C^*-algebras the latter property persists; since, at least in the prime case, it would yield a description of the norm of S.

This question was completely solved in [**5**].

Theorem 4.3. *For every C^*-algebra A, the following conditions are equivalent.*

(a) *For all $S \in \mathcal{E}\ell(A)$, $\|S\| = \|S\|_{cb}$;*
(b) *There is an exact sequence of C^*-algebras*

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

such that J is abelian and B is antiliminal.

C^*-algebras satisfying condition (b) of the theorem are called *antiliminal-by-abelian.* Let us remind the reader that a C^*-algebra A is said to be *antiliminal* if, for every non-zero positive element $a \in A$, the hereditary C^*-subalgebra $\overline{aAa}$ generated by a is non-abelian. As a consequence of the Glimm-Sakai theorem, for a dense set of irreducible representations π of an antiliminal C^*-algebra A, there are no non-zero compact operators contained in $\pi(A)$. In this sense, these are the 'Calkin algebra-like' C^*-algebras.

The antiliminal-by-abelian C^*-algebras had already appeared in connection with the study of factorial states in the work by Archbold and Batty [**4**] and will play an important role when we now turn to positivity properties.

5. Positivity Properties

In this section we will solely be concerned with C^*-algebras since they have an interesting order structure. Recall that an element a in a C^*-algebra A is said to be *positive* if $a = x^*x$ for some $x \in A$. Equivalently, a is self-adjoint and has positive spectrum. We denote the cone of all positive elements in A by A_+. A linear mapping $T\colon A \to B$ between two C^*-algebras is called *positive* if $TA_+ \subseteq B_+$.

In trying to characterise positive elementary operators it turns out that stronger positivity properties play an important role. This is due to the concept of matricial order on a C^*-algebra. Let $n \in \mathbb{N}$. The algebra $M_n(A)$ of $n \times n$ matrices over A is a C^*-algebra in a canonical, and unique, way. E.g., if $A \subseteq L(H)$ then $M_n(A) \subseteq L(H^n)$. Every linear mapping $T\colon A \to B$ can be extended to a linear mapping $T^{(n)}\colon M_n(A) \to M_n(B)$ by $T^{(n)}((x_{ij})) = (Tx_{ij})$ for each $(x_{ij}) \in M_n(A)$. If T is positive, then e.g. $T^{(2)}$ *need not be positive.* The simplest example is provided by the transposition on

$A = M_2(\mathbb{C})$. The mapping T is said to be *n-positive* if $T^{(n)}$ is positive. If T is n-positive for all $n \in \mathbb{N}$ then T is called *completely positive.*

In [**16**] we proved that $M_{a,b}$ is positive if and only if it is completely positive and then, it is of the form $M_{c^*,c}$ for some $c \in A$. This observation does not take over to more general elementary operators on arbitrary C^*-algebras. However, criteria for positivity of elementary operators can be given, see [**21**] and [**25**]. In the special case of the Calkin algebra once more the situation is very neat.

Theorem 5.1. [**13**, **21**] *Every positive elementary operator on $C(\ell^2)$ is completely positive.*

Let us sketch our argument for this result. Suppose $S = \sum_{j=1}^n M_{a_j,b_j}$ in $\mathcal{E}\ell(A)$ is positive. Then the question of whether S is completely positive can be reduced to the generic case that S is actually of the form $S = \sum_j M_{c_j^*,c_j} - M_{d^*,d}$ for some family of coefficients $\{c_j, d\}$ related to $\{a_j\}$, $\{b_j\}$. Now assume that d is not contained in the linear span of $\{c_j\}$. By a result of Magajna [**14**], there are then $x, y \in A = C(\ell^2)$ such that $xc_jy = 0$ for all j and $xdy \neq 0$. Since S is positive, we have $\sum_j c_j^*x^*xc_j \geq d^*x^*xd$. Hence, $\sum_j y^*c_j^*x^*xc_jy \geq y^*d^*x^*xdy$ which, by the choice of x and y, entails that $xdy = 0$, a contradiction. Therefore, d is a linear combination of $\{c_j\}$ and S can be re-written as $S = \sum_k M_{v_k^*,v_k}$, which proves complete positivity.

Since every positive operator on an abelian C^*-algebra is completely positive, we have two rather distant classes of C^*-algebras on which all positive elementary operators are completely positive. It turns out that these can be subsumed into the same class of C^*-algebras that occurred in the previous section.

Theorem 5.2. [**5**] *For every C^*-algebra A, the following conditions are equivalent.*

(a) *For all $S \in \mathcal{E}\ell(A)$, S positive $\implies$ S completely positive;*
(b) *A is antiliminal-by-abelian.*

The connection between this result and Theorem 4.3 becomes clearer if one realises that the cb-norm of an operator T is given by $\|T\|_{cb} = \sup_{n\in\mathbb{N}} \|T^{(n)}\|$ and hence relates to the matrix norm structure of a C^*-algebra algebra A.

The latter result has been extended by Timoney to cover the case of k-positive elementary operators [**27**]. He also made some recent contribution to the norm problem (see the previous section) in [**26**].

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