On a Problem of Finbarr Holland

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1. Introduction

1.1 In his article [H], Finbarr Holland has solved and discussed a problem that the students were given on the first morning of the 1999 IMO in Bucharest. They were asked to find, for all integers $n \geq 2$, the smallest constant C such that

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \Big(\sum_{i=1}^{n} x_i \Big)^4,$$

for all non-negative real numbers $x_1, x_2, \ldots, x_n$, and to specify when equality occurs for this value of C. At the end of [H], Finbarr Holland proposed the following problem.

1.2 Problem. *Let $n \geq 2$ be a positive integer and p a positive number. Determine* (a) *the best constant $C_{p,n}$ such that*

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^p + x_j^p) \leq C_{p,n} \Big(\sum_{i=1}^{n} x_i \Big)^{p+2}, \tag{1}$$

for all non-negative real numbers $x_1, x_2, \ldots, x_n$, and (b) *the cases of equality.*

The object of this note is to propose a solution to this problem. Precisely we shall prove the following the result.

1.3 Proposition. *Let $n \geq 2$ be a positive integer and p a positive real number. Then the following assertions hold true.*

(a) *The best constant $C_{p,n}$ satisfying* (1) *depends only on p and satisfies $\frac{1}{2^{p+1}} \leq C_{p,2} = C_{p,n}$ for all integers $n \geq 2$.*

(b) *For all $p > 0$, we have $C_{p,2} = \max\{(1-t)t^{p+1} + t(1-t)^{p+1} : 0 \leq t \leq \frac{1}{2}\}$.*

(c) *If $p \geq 2$, then $C_{p,2} = \frac{p^p}{(p+1)^{p+1}}$.*

(d) *If $1 < p < 2$, then $\frac{1}{2^{p+1}} \leq C_{p,2} \leq \frac{1}{2^{p+2}} + \frac{(p+1)^{p+1}}{(p+2)^{p+2}}$.*

(e) *If $0 < p \leq 1$, then $C_{p,2} = \frac{p^p}{(p+1)^{p+1}}$.*

2. Proofs

We start by giving an equivalent formulation of Problem 1.2.

2.1 Problem. *Let $n \geq 2$ be a positive integer and $p > 0$. Determine* (a) *the smallest constant $C_{p,n}$ such that*

$$\sum_{i=1}^{n} x_i^{p+1}(1-x_i) \leq C_{p,n},$$

for all non-negative real numbers $x_1, x_2, \ldots, x_n$ satisfying $\sum_{i=1}^{n} x_i = 1$, and (b) *the cases of equality.*

This formulation holds since we have the following equality:

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^p + x_j^p) = \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} x_i^{p+1}\right) - \sum_{i=1}^{n} x_i^{p+2}.$$

We let $\Delta_n := \{(x_1, x_2, \ldots, x_n) \in [0,1]^n : \sum_{i=1}^{n} x_i = 1\}$. For every real number t, we set $F_p(t) := (1-t)t^p$. For all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we put $H_p^n(x) := \sum_{i=1}^{n} x_i F_p(x_i)$. Another formulation of Problem 2.1 is then as follows.

2.2 Problem. *Let $n \geq 2$ be a positive integer and $p > 0$. Determine* (a) $\sup\{H_p^n(x) : x \in \Delta_n\} = C_{p,n}$, *and* (b) *the set of points in Δ_n where this maximum is attained.*

2.3. We start with the case $p \geq 1$. Along the lines of the paper [H], we distinguish four cases; Case (A), in which we suppose all of the x_i

are in the interval $[0, \frac{1}{2}[$; Case (B) in which we suppose that exactly one of the x_i belongs to $]\frac{1}{2}, 1]$; Case (C) in which we suppose that exactly one of the x_i equals to $\frac{1}{2}$; and Case (D) in which we suppose that precisely two of the x_i are equal to $\frac{1}{2}$. These cases exhaust all possibilities.

Case (A). It is easy to see that the function $x \mapsto x^p(1-x)$ is strictly increasing on the interval $[0, \frac{1}{2}[$, and that $\max\{x^p(1-x) : 0 \le x < \frac{1}{2}\} = \frac{1}{2^{p+1}}$. So, in this case, we have

$$\begin{aligned}\sum_{i=1}^{n} x_i^{p+1}(1-x_i) &\le \max_{1\le i\le n} x_i^p(1-x_i)\sum_{i=1}^{n} x_i \\ &= \max_{1\le i\le n} x_i^p(1-x_i) \\ &< \frac{1}{2^{p+1}}.\end{aligned}$$

Case (B). Suppose that x_k belongs to $]\frac{1}{2}, 1]$. We denote it by a and relabel the remaining $n-1$ variables as $y_1, y_2, \ldots, y_{n-1}$. We have the relation $1 - a = \sum_{i=1}^{n-1} y_i$, from which we deduce that each y_i belongs to $[0, 1-a] \subset [0, \frac{1}{2}[$. It follows that

$$y_i^p(1-y_i) \le a(1-a)^p,\ i = 1, 2, \ldots, n-1.$$

Hence

$$\begin{aligned}\sum_{i=1}^{n} x_i^{p+1}(1-x_i) &= a^{p+1}(1-a) + \sum_{i=1}^{n-1} y_i^{p+1}(1-y_i) \\ &\le a^{p+1}(1-a) + \max_{1\le i\le n-1} y_i^p(1-y_i)\sum_{i=1}^{n-1} y_i \\ &\le (1-a)a^{p+1} + a(1-a)^{p+1}.\end{aligned}$$

We conclude that in this case, we have

$$\sum_{i=1}^{n} x_i^{p+1}(1-x_i) \le \max\{(1-t)t^{p+1} + t(1-t)^{p+1} : \frac{1}{2} \le t \le 1\}.$$

Case (C). This is treated in a similar manner to Case (B), and we obtain

$$\sum_{i=1}^{n} x_i^{p+1}(1-x_i) \le \frac{1}{2^{p+1}}.$$

Case (D) is trivial, and it shows that we have $C_{p,n} \geq \frac{1}{2^{p+1}}$.

The conclusion of this analysis so far is that, if $x_1, x_2, \ldots, x_n$ are non-negative numbers satisfying $\sum_{i=1}^n x_i = 1$, then

$$\sum_{i=1}^{n} x_i^{p+1}(1-x_i) \leq \max\{(1-t)t^{p+1} + t(1-t)^{p+1} : \frac{1}{2} \leq t \leq 1\}.$$

It is clear that

$$C_{p,2} = \max\{(1-t)t^{p+1} + t(1-t)^{p+1} : \frac{1}{2} \leq t \leq 1\},$$

and it is easy to see that $C_{p,2}$ at most is

$$C_{p,n} = \max\left\{\sum_{i=1}^{n} x_i^{p+1}(1-x_i) : x_1, \ldots, x_n \in [0,1],\ \sum_{i=1}^{n} x_i = 1\right\}.$$

Thus, we have proved part of (a). The assertion (b) is clear.

2.4. We suppose that $p > 0$. Let us introduce the function

$$h_p(t) := H_p^2(t, 1-t) = t(1-t)[t^p + (1-t)^p] \quad \forall t \in \mathbb{R}.$$

Since the function F_p is concave on $[0, \infty[$ having $F_p(\frac{p}{p+1})$ as the (only) maximal value, we obtain, for all $t \in [0,1]$,

$$\begin{aligned} h_p(t) &= tF_p(t) + (1-t)F_p(t) \\ &\leq F_p(t^2 + (1-t)^2) \leq F_p(\frac{p}{p+1}) = \frac{p^p}{(p+1)^{p+1}}. \end{aligned}$$

Since $h_p(\frac{1}{2}) = \frac{1}{2^{p+1}}$, we deduce that $\frac{1}{2^{p+1}} \leq C_{p,2} \leq \frac{p^p}{(p+1)^{p+1}}$ for all $p > 0$.

2.5. We suppose that $p \in [2, \infty[$. We shall prove that $C_{p,2} = \frac{p^p}{(p+1)^{p+1}}$. To this end, we consider $u := \frac{1}{2}(1 - \sqrt{\frac{p-1}{p+1}})$ and $v := \frac{1}{2}(1 + \sqrt{\frac{p-1}{p+1}})$; these are the solutions (belonging to [0,1]) of the algebraic equation $t^2 + (1-t)^2 = \frac{p}{p+1}$. According to the convexity of the function $t \mapsto t^{p-1}$ on $[0, \infty[$, we deduce that

$$\begin{aligned} h_p(u) &= \frac{p}{2(p+1)}(uu^{p-1} + vv^{p-1}) \\ &\geq \frac{p}{2(p+1)}(u^2 + v^2)^{p-1} = \frac{p^p}{2(p+1)^p} \geq \frac{p^p}{(p+1)^{p+1}}. \end{aligned}$$

According to 2.5, we conclude that $C_{p,2} = \frac{p^p}{(p+1)^{p+1}}$.. Thus we have proved (c). The assertion (d) is easy to obtain, hence we omit it.

2.6. To complete the proof, we look at the case $0 < p \leq 1$. It is clear that in this case, the function $t \mapsto t^{p-1}$ is decreasing and convex on $]0, \infty[$. For all $t \in]0, \frac{p}{(p+1)}]$, we have $1 - t \geq \frac{1}{p+1}$ and $t^2 + (1-t)^2 \geq \frac{1}{2} \geq \frac{p}{p+1}$, therefore

$$\begin{aligned} C_{p,2} &\geq h_p(t) = t(1-t)(tt^{p-1} + (1-t)(1-t)^{p-1}) \\ &\geq t(1-t)(t^2 + (1-t)^2)^{p-1} \geq t\frac{p^{p-1}}{(p+1)^p}. \end{aligned}$$

We deduce then that $C_{p,2} \geq \frac{p^p}{(p+1)^{p+1}}$. For every integer $n \geq 2$ and all $x \in \Delta$, we have

$$\sum_{i=1}^{n} x_i^{p+1}(1 - x_i) \leq F_p(\frac{p}{p+1}) \sum_{i=1}^{n} x_i = F_p(\frac{p}{p+1}).$$

Thus, we find $C_{p,2} = C_{p,n} = F_p(\frac{p}{p+1}) = \frac{p^p}{(p+1)^{p+1}}$. So we have proved Proposition 1.3.

3. Remarks

3.1. In the case where $1 < p < 2$, we have only obtained upper and lower estimates for the best constant $C_{p,n}$. We believe that (in this case) the following equality holds

$$C_{p,2} = F_p(\frac{p}{p+1}),$$

but we have no proof. We conclude this note by giving another problem generalizing Problem 1.2.

3.2 Problem. *Let n, k be positive integers such that $k \leq n$ and $p > 0$. Determine* (a) *the smallest constant $C_{p,n,k}$ such that*

$$\sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} x_{i_1} x_{i_2} \ldots x_{i_k} \, (x_{i_1}^p + x_{i_2}^p + \ldots + x_{i_k}^p) \leq C_{p,n,k} \Big(\sum_{i=1}^{n} x_i \Big)^{p+k},$$

for all non-negative real numbers $x_1, x_2, \ldots, x_n$ satisfying $\sum_{i=1}^n x_i = 1$, and (b) *the cases of equality.*

References

[H] Finbarr Holland, Three problems in one, *Irish Mathematical Society Bulletin* **43** (1999), 73–78.

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