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A SURVEY OF SUBNORMAL SUBGROUPS

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Introduction

Since the appearance of Helmut Wielandt's fundamental paper [27] over fifty years ago, much progress has been made in the theory of subnormal subgroups thanks to the contribution of many distinguished group theorists.

A comprehensive and masterly exposition of the theory of subnormal subgroups is due to Lennox and Stonehewer. The purpose of this article, based on a talk given at "Groups in Galway" is to present some of the remarkable results in the theory without encumbering the general reader with technical details or proofs. The selection of topics is not exhaustive and reflects a bias of the author, but it is hoped to whet the appetite of the reader, who is referred to Lennox and Stonehewer [12] in the first instance. Notation is standard and follows that of Lennox and Stonehewer [12] or Robinson [22].

Definition. *If H is a subgroup of a group G such that*

$$x^{-1}Hx = H \quad \forall x \in G$$

then H is normal in G, written $H \triangleleft G$.

If $L \triangleleft H$ and $H \triangleleft G$ it does not follow that $L \triangleleft G$, i.e. for subgroups of a group normality is not a transitive relation, as can be verified by examining the alternating group on 4 letters, $\mathcal{A}_4$, for instance. This may serve as motivation for the following relation on subgroups which *is* transitive:

Definition. *A subgroup H is subnormal in a group G if H occurs as a term in a finite normal series*

$$H = H_m \lhd H_{m-1} \lhd \cdots \lhd H_0 = G \qquad (*)$$

from H to G, where $H_i \lhd H_{i-1}$ for each i.

Notation: $H \lhd\lhd G$ or also H *sn* G will mean H is subnormal in G.

Definition. *The length of the shortest normal series from H to G is called the defect of H in G, written $def(G, H)$.*

Definition. *The normal closure of H in G, H^G, is $\langle H^g | g \in G\rangle$, the group generated by all the conjugates of the subgroup $H (H^g := g^{-1}Hg)$, and this is of course a normal subgroup of G. It is the smallest normal subgroup of G which contains H.*

Thus if H is subnormal of defect m in G, we see that

$$H^G \leq H_1 \lhd H_0 = G.$$

Replacing G by H_1 we get

$$H^{H_1} \leq H_2 \lhd H_1$$

Iterating this process, we see that by taking successive normal closures, in at most m steps the process terminates with arrival at $H_m = H$.

Notation: Put $H_{(0)} = G$, set $H_{(i+1)} = H^{H_{(i)}}$ then $H_{(1)} = H^G \leq H_1$ (as in (*) above) $H_{(2)} = H^{H_{(1)}} \leq H^{H_1} \leq H_2$ and in general $H_{(i)} \leq H_i$, the *ith* term in the normal series (*).

The series $H_{(i)}$ is the most rapidly descending series from G to H, and $H \lhd\lhd G$ if and only if $H = H_{(i)}$ for some $i \geq 0$.

Examples:

(i) All normal subgroups of a group are subnormal of defect 1.

(ii) Subgroups of order 2 in $\mathcal{A}_4$ have defect 2 in $\mathcal{A}_4$.

(iii) In $D_{2^m} \simeq \langle a, b | a^{2^{m-1}} = b^2 = 1, b^{-1}ab = a^{-1}\rangle, m \geq 3$, the non-central subgroups of order 2 have defect precisely $m - 1$.

Some elementary facts concerning subnormal subgroups are:
(i) Transitivity: If $H \lhd\lhd K$ and $K \lhd\lhd G$ then $H \lhd\lhd G$.
(ii) If $H \lhd\lhd G$ and θ is a homomorphism of G onto G^θ, then $H^\theta \lhd\lhd G^\theta$, moreover $def(G^\theta, H^\theta) \leq def(G, H)$.
(iii) If $H \lhd\lhd G$ and K is a subgroup of G then

$$H \cap K \lhd\lhd K \quad \text{and} \quad def(K, H \cap K) \leq def(G, H)$$

(iv) If $H \lhd\lhd G$ then $N_G(H)$, the normalizer of H in G, $\{g | g^{-1}Hg = H\}$ is strictly larger than H i.e. $N_G(H) > H$ (the converse of this fact is false in general). Of course $N_G(H)$ is the largest subgroup of G which contains H as a *normal* subgroup.
(v) If each member of a (finite) collection of subgroups H_j is subnormal of defect at most k in G, then

$$\bigcap_j H_j$$

is also subnormal of defect at most k in G. (If the defects of the H_j are not bounded then an example on p.373 of Robinson [22] shows that (v) is false without this condition.)

Some of these facts (ii), (iii), (iv), (v) are valid also for normal subgroups.

On the other hand, whereas normal subgroups have the property of permuting with elements of G ($N \lhd G$, then $Ng = gN$) and thus with subgroups of G ($N \lhd G, H \leq G$ then $NH = HN$), this is not usually the case for subnormal subgroups. Another fact concerning two normal subgroups N_1, N_2 of a group G is that N_1N_2 is also normal in G. This leads to the earliest but most famous question concerning subnormal subgroups.

The Join Problem

Let H, K be subnormal in G. Under what circumstances will $\langle H, K\rangle$—the join of H and K—be subnormal in G? This problem has been the subject of intense research, starting with Wielandt's fundamental paper in 1939 [27] and culminating with a remarkable result of J.P. Williams [32].

In 1939 Wielandt proved that if G is finite and H, K are subnormal in G then $\langle H, K\rangle$ is also subnormal in G. Thus the set of subnormal subgroups of a finite group G forms a sublattice of the lattice of all subgroups of a finite group G. In fact Wielandt showed the result was true provided G satisfied the maximal condition for subnormal subgroups *max-sn* whereby every strictly ascending chain of subnormal subgroups of G has finite length, (so every non–empty set of subnormal subgroups of G contains a maximal member). In the case of G finite there is an elegant proof due to Kegel, using induction on the order of G, see [12] p.8.

A criterion to guarantee subnormality of $\langle H, K\rangle$ when $H, K \lhd\lhd G$ is that K should normalize H i.e. $K \leq N_G(H)$ (**) One shows that K normalizes $H_{(i)}$ and since $K \lhd\lhd H_{(i-1)}K$ and $H_i \lhd H_{(i-1)}K$ one obtains $H_{(i)}K \lhd\lhd H_{(i-1)}K$ with defect $\leq def(G, K)$. In fact $def(G, J) \leq def(G, H)\cdot def(G, K)$. The next result is useful also.

Lemma. *Let $H, K \lhd\lhd G$. Put $J = \langle H, K\rangle$ then the following are equivalent:*
(i) $J \lhd\lhd G$,
(ii) $H^K (= \langle H^k | k \in K\rangle) \lhd\lhd G$ and
(iii) $[H, K](= \langle [h, k](= h^{-1}k^{-1}hk), h \in H, k \in K\rangle) \lhd\lhd G$.

Since $[H, K] \lhd H^K \lhd J$ one only has to show (iii) $\Rightarrow$ (i). (See [12] p.4.)

If G is a nilpotent group of class c, that is, if

$$[\underbrace{G, G, \ldots, G}_{c+1 G's}] = \langle 1\rangle$$

then every subgroup of G has defect at most c. In particular if $G' = [G, G]$ is nilpotent then from part (iii) of the Lemma above it follows that for any subnormal subgroups H, K of G their join is also subnormal. A generalization of Wielandt's result is due to Robinson [19].

Theorem (Robinson). *Let $H, K \lhd\lhd G$ and suppose G' satisfies $max - sn$. Then $J := \langle H, K\rangle \lhd\lhd G$.*

Another easy criterion—a companion to (**)—is the following:

Lemma. *Let $H, K \lhd\lhd G$ and put $J = \langle H, K\rangle$. If $HK = KH$ then $J \lhd\lhd G$.*

(Of course, $KH = HK$ does not imply that $K \leq N_G(H)$.)

So it is clearly of interest to the join problem to determine conditions under which H and K permute. (If J equals HKH then in fact J equals HK, but a counterexample due to R.S. Dark [12] p.20 shows that one can have $H, K \lhd\lhd G, J = HKHK$ but J not subnormal in G!) A famous permutability criterion is due to Roseblade [23].

Theorem (Roseblade). *If H and K are subnormal subgroups of a group G such that the tensor product of the abelian groups (regarded as $\mathbf{Z}$-modules)*

$$H/H' \bigotimes K/K'$$

is trivial (one says H is orthogonal to K, written $H \perp K$) then $HK = KH$, and thus $\langle H, K\rangle \lhd\lhd G$. Moreover if H and K are not orthogonal then there exists a group G_0 such that $H \simeq H_0 \lhd\lhd G_0, K \simeq K_0 \lhd\lhd G_0$ and $H_0K_0 \neq K_0H_0$.

In 1958, Zassenhaus [33] published an example [Exercise 23, Appendix D in his book "Theory of Groups"] showing that the join of two subnormal subgroups could fail to be subnormal. The group G constructed by Zassenhaus consisted of a module with a specially defined basis, over $\mathbf{Z}$, extended by suitably chosen automorphisms. This group was countable and abelian by nilpotent of class 2 i.e. G/A was nilpotent of class 2, with A abelian. Two subnormal subgroups H, K each had defect 3 in G and their join $\langle H, K\rangle$ was nilpotent of class 2 but not subnormal in G since one shows $J^G = G$. (See also Robinson [22] p.375.) It is worth remarking that in an example H cannot have defect 2 since then $H \lhd H^G \lhd G$ and so $H^K \lhd H^G \lhd G$, K normalizes H^K and one would get $J = \langle H, K\rangle \lhd\lhd G$. So in an example the defect of $H(K)$ must be at least three which is the case in the Zassenhaus example. Also G' is not abelian and neither is J. Thus in some respects Zassenhaus' example is the minimum one can get away with!

To conclude with a necessary *and* sufficient criterion to ensure that the join of a pair of subnormal subgroups is subnormal, the following is a result of J.P. Williams [32]:

Theorem (Williams). *Let H, K be groups:*
(i) If $H/H' \otimes K/K'$ (as an abelian group) is the (direct) sum of a group U of finite rank and a periodic divisible group V (c.f. Robinson [22] pp. 94–97) then $\langle H, K\rangle$ is subnormal in all groups in which H and K can be subnormally embedded.
(ii) Conversely if $H/H' \otimes K/K'$ does not have the structure in (i) as an abelian group, then there is a group G containing H, K as subnormal subgroups such that $\langle H, K\rangle$ is not subnormal in G.

The proof of this theorem (which is the subject of chapter 5 in [12]) involves extensive development of ring-theoretic machinery first introduced by Philip Hall in the 1950's.

The Wielandt Subgroup

In [1] Baer defined the "Kern" of a group as the intersection

$$\bigcap_{H \leq G} N_G(H)$$

of the normalizers of all the subgroups of G. In 1958, Wielandt [28] considered an analogous intersection

$$w(G) = \bigcap_{S \lhd\lhd G} N_G(S)$$

i.e. the intersection of the normalizers of all the subnormal subgroups of G.

Whereas $w(G)$ may equal $\langle 1\rangle$ as in the case of $G = D_\infty$ the infinite dihedral group, Wielandt proved the following rather surprising results [28].

Theorem (Wielandt). *If $|G|$ is finite then $w(G) \neq \langle 1\rangle$.*

Theorem (Wielandt). *Let G be an arbitrary group. Then $w(G)$ contains*
(i) every simple non–abelian subnormal subgroup of G and

(ii) every minimal normal subgroup M of G where M satisfies the minimal condition for subnormal subgroups. ($min - sn$).
[Indeed, if G satisfies $min - sn$ then Robinson [20] has shown that $|G : w(G)|$ is finite.]

If G is a finite group, then $w(G) \neq \langle 1\rangle$. Consequently

$$w(G/w(G)) \neq \langle 1\rangle.$$

Setting $w_0(G)) = \langle 1\rangle$ and $w_{i+1}(G)/w_i(G) := w(G/w_i(G))$, for some finite i one will have $w_i(G) = G$. The smallest such i is called the *Wielandt length* of the group G.

The Wielandt subgroup has been the subject of a paper by Camina [4] in which he investigates relations between the Wielandt length, derived length and Fitting length for a finite soluble group G. This work has been improved by Bryce and Cossey [3] who obtain best possible bounds for both the derived and Fitting length of a finite soluble group in terms of its Wielandt length. Casolo [8] has extended these results to infinite soluble groups of finite Wielandt length. Another result concerning the Wielandt subgroup due to Brandl, Franciosi and de Giovanni [2] is the following

Theorem. *Let G be a polycyclic group (G has a normal series with each factor cyclic) which is either*
(a) metanilpotent (an extension of a nilpotent group by a nilpotent group) or
(b) abelian by finite. Then $w(G)/Z(G)$ is finite.

The Wielandt subgroup also has the property that since $w(G) \lhd G$, a subnormal subgroup K of $w(G)$ is subnormal in G, hence $N_G(K) \geq w(G)$ thus $K \lhd w(G)$. In other words, $w(G)$ is a group in which normality *is* a transitive relation. Such groups are called T-groups and for groups in this class $G = w(G)$, and all subnormal subgroups of G have defect 1. Finite soluble T–groups have been classified by Gaschütz [9], and Robinson [18] has shown that in fact every soluble T–group is metabelian.

Groups with every subgroup subnormal

Of course, if G is an abelian group, then every subgroup of G is subnormal. More interesting is the case of non–abelian groups with every subgroup subnormal.

Suppose first G is a non–abelian group with every subgroup **normal**, then G is a non–abelian Dedekind group and the structure of G is described in [Robinson, [22] p.139]; for instance Q_8 is an example of a non–abelian Dedekind group.

In view of the fact that if G is a nilpotent group of class c then every subgroup of G is subnormal with defect at most c, Roseblade [24] was able to show that if G is a group in which every subgroup is subnormal of defect at most d then there is a function $f(d)$ such that G is nilpotent of class at most $f(d)$. A specific result of Heineken [10] and Mahdahvianary [14] in this area is the following: If all cyclic subgroups of G have defect at most 2, then G is nilpotent of class 3.

However, a celebrated example due to Heineken & Mohamed [11] shows that there are groups G in which every subgroup is subnormal (but there are no bounds on the defects) and $Z(G) = 1$, so G is not even hypercentral. Moreover, for $H < G$ one has $N_G(H) > H$, i.e. G satisfies the so–called *normalizer condition*, whereby every proper subgroup of G is a proper subgroup of its normalizer.

Casolo [6] has shown that if G is a group with every subgroup subnormal, then for some n, $G^{(n)} = G^{(n+1)}$, i.e. the derived series breaks off after finitely many terms. Recently Möhres [16] proved that a group G with every subgroup subnormal is in fact soluble. It would appear that groups in the class of groups with every subgroup subnormal, are in fact metanilpotent.

The class of B_n groups G in which subnormal subgroups have defect at most n has been investigated. B_1 groups are the aforementioned T–groups, in which normality is a transitive relation. Since simple groups trivially are B_1–groups, one restricts attention to soluble B_n–groups. An interesting problem is to try and bound (if possible) the derived lengths of soluble B_n–groups in terms of n. Examples due to Robinson [21], show that even in the class of B_2–groups all derived lengths can occur (using a construction based on iterated wreath products) but these groups are torsion free. In the case of periodic soluble B_2–groups Casolo [5] has shown that they have derived length at most 10. In the case of finite soluble B_2–groups Casolo has shown that they have derived length at most 5 and Fitting length at most 4. An example of McCaughan and Stonehewer [13] shows that this last result of Casolo is best possible. Thus there is much scope for investigating the interrelationship between the derived length and Fitting length of soluble periodic B_n–groups.

Criteria for Subnormality

Let H be a subgroup of G. Suppose that $HK = KH$ for any subgroup K of G, we say H is a *permutable* subgroup of G or H is *quasinormal* in G, written H *per* G. Clearly normal subgroups are permutable. Not every subnormal subgroup is permutable (one can verify this by examining A_4). Not every permutable subgroup is normal, if $G = \langle a, b | a^8 = b^2 = 1, b^{-1}ab = a^5 \rangle$ then $\langle b \rangle$ *per* G, but $\langle b \rangle$ is not normal in G. Ore [17] showed that a maximal permutable subgroup of a group G is normal in G and as a corollary one obtains that a permutable subgroup H of a finite group G is subnormal in G. This corollary has been extended by Stonehewer [25] to finitely generated groups.

A more restrictive form of permutability is that of permuting with conjugates i.e. $VV^g = V^gV$ for all $g \in G$. A result of Ore [17] and Szép [26], is that in a finite group G such a subgroup V which permutes with its conjugates is subnormal in G. Wielandt [29] has considered similar criteria and the following Theorem is due to him.

Theorem (Wielandt). *Let G be a finite group and A, B subgroups of G such that*

$$AB^x = B^xA \quad \text{for all} \quad x \in G.$$

Then
(i) If $G = AB^G = BA^G$ then $G = AB$.
(ii) $A^B \cap B^A \lhd\lhd G$.
(iii) If $AB \leq H \leq G$ then $A^H \cap B^H \lhd\lhd G$.
(iv) If X, Y are subsets of G then $[A^X, B^Y] \lhd\lhd G$.

Another result of Wielandt's follows: First note that if $N \lhd G$, then (i) $N \lhd \langle N, g\rangle \forall g \in G$ and (ii) $[n, g] \in N \forall g \in G$. Moreover the converse of (i) or (ii) implies $N \lhd G$. If $H \lhd\lhd G$ then clearly (i)$'$ $H \lhd\lhd \langle H, g\rangle \forall g \in G$ and (ii)$'$ for any $g \in G$ and some positive integer n and $h \in H, [g, \underbrace{h, \ldots, h}_{n\ h's}] \in H$. Wielandt [30] has shown that for finite groups (i)$'$ and (ii)$'$ are each sufficient for H to be subnormal in G.

Theorem (Wielandt). *Suppose $H \leq G$ and G is finite. Then the following are equivalent to $H \lhd\lhd G$:*
(i) $H \lhd\lhd \langle H, g\rangle \quad \forall g \in G$
(ii) $H \lhd\lhd \langle H, H^g\rangle \quad \forall g \in G$.
(iii) $H \lhd\lhd \langle H, H^{h^g}\rangle \quad \forall h \in H, g \in G$.

To conclude our survey we mention the **subnormalizer** of a subgroup. The normalizer of a subgroup H in a group G, $N_G(H) = \{g \in G | H \lhd \langle H, g\rangle\}$, and $H \lhd G \Leftrightarrow N_G(H) = G$. Consider the following:

Definition: Denote by $S_G(H) = \{g \in G | H \lhd\lhd \langle H, g\rangle\}$ the *subnormalizer* (in G) of the subgroup H.

One must note that in general $S_G(H)$ is not a subgroup! Take $H = \langle (12)(34), (13)(24), (14)(23), (12)\rangle$ and $K = \langle (23)(45), (24)(35), (25)(34), (25)\rangle$. Then $|H| = |K| = 8$ and they are Sylow 2-subgroups of $\mathcal{S}_5$, the symmetric group of degree 5. $|H \cap K| = 2$ and $H \cap K = \langle (34)\rangle$ is not subnormal in $\mathcal{S}_5$. Of course $H \cap K \lhd\lhd H, K$. If the subnormalizer of $H \cap K$ were a subgroup, it would contain H and K and hence would contain $\langle H, K\rangle$ which is $\mathcal{S}_5$, a contradiction, because $H \cap K$ is not subnormal in $\mathcal{S}_5$. In this example $HK \neq KH$ as subsets. The following result shows when the subnormalizer *is* a subgroup.

Theorem (Maier [15], Wielandt [31]). *Suppose G is a finite group and $G = AB$ with $A, B \leq G$. If $H \lhd\lhd A$ and $H \lhd\lhd B$ then $H \lhd\lhd G$.*

Thus in a finite group G whenever $H \lhd\lhd U, H \lhd\lhd V (H, U, V$ subgroups of $G)$ implies $H \lhd\lhd \langle U, V\rangle$ then $S_G(H)$ *is* a subgroup. Wielandt [31] has formulated a number of conjectures regarding

criteria for subnormality of a subgroup H in the finite group G where $G = AB$ for subgroups A and B.

The Class of s_n–groups

Call G an s_n–group if the subnormalizer of every subgroup of G is itself a subgroup. For any element x in a group G, denote by $E_G(x) = \{g \in G | [g, \underbrace{x, \ldots, x}_{n\ x's}] = 1 \text{ for some } n \in \mathbb{N}\}$. If we denote by E–groups the class of groups in which $E_G(x)$ is a subgroup for every x in G then a recent result due to Casolo [7] is the following:

Theorem (Casolo). *Let G be a finite group. Then G is an s_n–group if and only if G is an E–group.*

In addition, Casolo has proved that a finite group G is an s_n–group if and only if the intersection of any two Sylow subgroups of G is pronormal in G, whereby a subgroup H is pronormal in G if H is conjugate to H^g in $\langle H, H^g\rangle$. Thus as the reader can see, there are new areas of investigation in the theory of subnormal subgroups which yield surprising connexions with other topics in group theory such as Engel elements or pronormality.

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