We can now state Martin's axiom (MA):

Suppose $(P, \leq)$ *is a partially ordered set satisfying the countable chain condition. If* (D_i) $(i \in I)$ *is a family of dense subsets of* P *with* $|I| < 2^{\aleph_0}$, *then there is a compatible subset* G *such that* $G \cap D_i \neq \emptyset$ *for all* $i \in I$.

An observant naive set theorist will notice that MA follows from the continuum hypothesis (CH). However it has also been shown that (ZFC + MA + negation of CH) is *consistent*. (By consistent we mean that if ZFC is free from contradictions then so also is the above.) Indeed it is also known that ZFC + (V=L) is consistent.

Shelah's answer to the Whitehead problem was this: In (ZFC + MA + negation of CH) there is a group A (of cardinality $\aleph_1$) which satisfies the conditions of Whitehead's problem but A is not free.

The outcome is, of course, that for naive set theorists the problem is undecidable! This of course was a considerable shock to most people working in Abelian groups. (See Eklof [3] for a very readable discussion of this area.)

RECENT DEVELOPMENTS

While the Whitehead Problem is of no direct importance for the Realization Problem, the techniques developed by Shelah in his 1974 paper (and subsequently extended by him) have become the major tool for tackling the problem. The following results indicate some of the many recent advances made:

1. (ZFC + (V=L)). Every cotorsion-free ring is an endomorphism ring. (Dugas and Gobel, 1981).

   (A group is cotorsion-free if it is torsion-free, reduced and contains no copy of J_p, for any p.)

2. (ZFC). If A is any algebra over a complete discrete valuation ring R then there exists a R-module G having A as its "essential" endomorphism ring (Dugas, Gobel and Goldsmith, 1982).

3. (ZFC). Every cotorsion-free algebra is an endomorphism algebra (Dugas and Gobel, 1982).

The state of the art for the Realization Problem (in 1984) has been very elegantly presented in a unified approach by Corner and Gobel [2]. Their results are based on a combinational technique devised by Shelah. In very recent work, Dugas and Gobel and Gobel and Goldsmith have established (in V = L) that most realizations can be obtained in classes of groups which are almost free (in the sense that all subgroups of cardinality less than the cardinal of the realizing group are free). Some of the results so obtained are undecidable in ZFC.

CONCLUDING REMARKS

One of the principal objectives in writing this paper is to convince non-logicians that set and model theory will have a role in our subjects once we deal with any uncountable structure. (Since $\mathbb{R}$ is uncountable that takes in most of us!) This impact is perhaps most apparent in Abelian group theory but the reason for this is clear - finite Abelian groups are completely classified being direct sums of cyclic groups. However other areas of algebra, topology and analysis will slowly but surely become involved also.

REFERENCES

1. CORNER, A.L.S.
   "Every Countable Reduced Torsion-Free Ring is an Endomorphism Ring", *Proc. London Math. Soc.*, (3) 13 (1963) 867-710.

n is itself prime).

So, for example, $\mu(4) = 0$, $\mu(6) = 1$, $\mu(7) = -1$, $\mu(42) = -1$.

The function $M(n)$ is then defined as

$$M(n) = \sum_{k=1}^{n} \mu(k).$$

The following table gives the first twenty values of this function:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
M(n)	1	0	-1	-2	-1	-2	-2	-2	-2	-1	-2	-2	-3	-2	-1	-1	-2	-2	-3	-3

If you were to continue the above table out into the hundreds or thousands, you would discover that the behaviour of the function $M(n)$ is quite erratic, fluctuating wildly from positive to negative. But, the value of $|M(n)|$ always appears to be less than $\sqrt{n}$, i.e.

$$|M(n)| < \sqrt{n} \quad (1)$$

for all values of n. Since the Riemann Hypothesis follows (fairly easily) from any universally valid inequality of the form

$$|M(n)| < A\sqrt{n} \quad (2)$$

for A a constant, Stieltjes' claim in his letter to Hermite to have proved inequality (2) for some A, would have, if true, resolved Riemann's problem at once. It was because of this claim of Stieltjes that when Hadamard wrote his now classic and greatly acclaimed paper proving the Prime Number Theorem in 1896, he apologised for publishing a proof of an already established result. (It was known that the Prime Number Theorem follows from the Riemann Hypothesis.) In fact, as we now know, Stieltjes was in all probability wrong in his claim, and his failure to ever produce a proof may indicate that he himself realised his error. But as will become clear,

it has taken the most powerful computing machinery available in 1985 to settle this matter of inequality (1) conclusively, *and even then no one has produced a specific counter example to the inequality.* Inequality (2) for $A > 1.06$ remains unsolved!

The first systematic investigation of the problem by computational means was in 1897 when F. Mertens produced (by hand calculation) a 50-page table of selected values of $\mu(n)$ and $M(n)$ for n up to 10,000. Since all his tabulated values satisfied inequality (1), he concluded that the inequality was indeed 'very probable'. Though his conclusion was wrong, it was this work which led to his name being attached to the conjecture. The *Mertens Conjecture* is the assertion that inequality (1) is valid for all values of n. (Stieltjes himself had also conjectured that the constant A in his claimed inequality (2) could be taken to be 1.)

The considerable computational evidence obtained subsequent to Mertens' work all tended to support the conjecture. In a series of papers between 1897 and 1913, R.D. von Sterneck published additional values of $M(n)$ for n up to 5×10^6, and in 1963, G. Neubauer computed all values for n less than 10^8, and selected further values for n up to 10^{10}. In 1979, M. Youinaga reached 4×10^8. All these values satisfied not only the Mertens inequality (1), but the even stronger

$$|M(n)| < 0.6\sqrt{n}. \quad (3)$$

The first value of n for which $|M(n)| \geq 0.5\sqrt{n}$ is n=7,725,038,629, when $M(n) = 43,947$. This was obtained by Cohen and Dress in 1979, who calculated $M(n)$ for all n up to 7.8×10^9. But even they did not find any value for n for which inequality (3) is violated. Thus the numerical evidence in support of the Mertens Conjecture is quite strong - at least it seems so to the human mathematician used to dealing with much smaller numbers. (In point of fact the numerical evidence in support of the conjecture had been 'discounted' long before the final disproof was obtained: analytical evidence pointed the other way.)

The first step in attacking the Mertens Conjecture analytically involves regarding the function M as defined not just on the natural numbers but on all non-negative reals x. To do this simply let $M(x) = M([x])$, where $[x]$ is the largest integer not greater than x (with $M(x) = 0$ if $x < 1$). Inequality (1) can now be re-written as (for all $x \geq 0$)

$$|M(x)|x^{-\frac{1}{2}} < 1 \qquad (4)$$

and inequality (2) as (for all $x \geq 0$)

$$|M(x)|x^{-\frac{1}{2}} < A \qquad (5)$$

Stieltjes claimed to have proved (5), and conjectured (4). Mertens thought that (4) was 'probable'. The present day conjecture is that, on the contrary

$$\limsup_{x\to\infty} |M(x)|x^{-\frac{1}{2}} = \infty. \qquad (6)$$

(This remains, however, an *unproved* conjecture.)

The result of te Riele and Odlyzko which disproves the Mertens Conjecture is that

$$\limsup_{x\to\infty} |M(x)|x^{-\frac{1}{2}} > 1.06 \qquad (7)$$

(No single x is produced for which $|M(x)|x^{-\frac{1}{2}} > 1.06$. The proof is indirect. The discoverers conjecture that no such x exists below at least 10^{20}.)

The connection with the Riemann Hypothesis is quite easily verified. If $\zeta(s)$ is the Riemann zeta function, then for $Re(s) > 1$ we have

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

and some manipulation gives (for $Re(s) > 1$):

$$\frac{1}{\zeta(s)} = s\int_1^{\infty} \frac{M(x)}{x^{s+1}}\,dx.$$

Since $M(x)$ is constant on each interval $[n,n+1)$, if inequality (4) (or (5)) held, then the integral in this last identity would define a function analytic in $Re(s) > \frac{1}{2}$, which would give an analytic continuation of $1/\zeta(s)$ to $Re(s) > \frac{1}{2}$. But then the function $\zeta(s)$ would have no zeros in $Re(s) > \frac{1}{2}$, which is the statement of the Riemann Hypothesis.

In fact the Riemann Hypothesis is probably equivalent to

$$|M(x)| = O(x^{\frac{1}{2}+\varepsilon})$$

for all $\varepsilon > 0$. (This was known to Stieltjes.)

Turning now to the specific problem of disproving the Mertens Conjecture by establishing an inequality such as (7), we begin by setting

$$x = e^y, \qquad -\infty < y < \infty.$$

Now define

$$m(y) = M(x)x^{-\frac{1}{2}} = M(e^y)e^{-y/2}.$$

The aim then is to prove that

$$\limsup_{y\to\infty} m(y) > 1.06$$

(or indeed any such inequality where the right hand side is greater than 1.)

The crux of the argument now is to define a function $h(y)$ such that

(i) for any y_0, $\limsup_{y\to\infty} m(y) \geq h(y_0)$;

(ii) it is possible to compute (in practice!) values of h;

(iii) a y_0 can be found for which $h(y_0) > 1$.

(This approach goes back to work of A.E. Ingham in 1942.)

Without going into any details, (i) and (ii) are achieved by means of the following theorem. (Just skip over this part if there are concepts unfamiliar to you.)

THEOREM Suppose that $K(y) \in C^2(-\infty,\infty)$, $K(y) \geq 0$, $K(-y) = K(y)$, $K(y) = O((1+y^2)^{-1})$ as $y \to \infty$; and suppose further that if $k(t)$ is defined by

$$k(t) = \int_{-\infty}^{\infty} K(y)e^{-ity}dy,$$

If the then $k(t) = 0$ for $|t| \geq T$ for some T, and $k(0) = 1$. zeros $\rho = \beta + i\gamma$ of the zeta function with $0 < \beta < 1$ and $|\gamma| < T$ satisfy $\beta = \frac{1}{2}$ and are simple, then for any y_0,

$$\limsup_{y\to\infty} m(y) \geq h_K(y_0),$$

where

$$h_K(y) = \sum_{\rho} k(\gamma)\frac{e^{iy\gamma}}{\rho\zeta'(\rho)}.$$

The simplest function which satisfies the conditions of the above theorem is

$$k(t) = \begin{cases} 1 - \frac{|t|}{T}, & |t| \leq T \\ 0, & |t| > T. \end{cases}$$

Using such a $k(t)$ with $T = 1{,}000$, Spira (1966) showed that

$$\limsup_{y\to\infty} m(y) \geq 0.5355.$$

The function used by te Riele and Odlyzko is

$$K(t) = g(\tfrac{t}{T}),$$

where $T = 2515.286\ldots$ is the height of the 2,000th zero of the zeta function and

$$g(t) = \begin{cases} (1-|t|)\cos(\pi t) + \pi^{-1}\sin(\pi|t|), & |t| \leq 1 \\ 0, & |t| \geq 1. \end{cases}$$

Finding the value of y_0 for which the corresponding function $h_K(y_0)$ is greater than 1 then involves an accurate (100 decimal digits) computation of the first 2,000 zeros of the zeta function. This was done using a Newton process and took some 40 hours of CPU time on a CDC CYBER 750 computer at SARA (the Amsterdam Computer Centre). With these values available, finding the required y_0 was achieved using a new algorithm for diophantine approximation due to Lenstra, Lenstra and Lovasz (1982) and took about 10 hours of CPU time on a CRAY-1 computer at Bell Laboratories in Murray Hill, New Jersey. (As you might imagine, the method was not a 'blind search'. Indeed the function $h_K(y)$ only 'rarely' gives a value greater than 0.5, let alone greater than 1.)

The 'magic value of y_0 that the computer found is a negative number of the order 1.4×10^{65}. For this y_0, $h_K(y_0) = 1.061$ (to three decimal places). The 'exact' values are quoted in the paper referenced below.

REFERENCE

For a much fuller account of the solution to the Mertens problem, together with an extensive bibliography on the problem, see *Disproof of the Mertens Conjecture* by A.M. Odlyzka and H.J.J. te Riele, *Journal fur die Reine und Angewandte Mathematik*, 357 (1985) pp. 138-160.

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